Convergence Acceleration via Combined Nonlinear-Condensation Transformations

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A method of numerically evaluating slowly convergent monotone series is described. First, we apply a condensation transformation due to Van Wijngaarden to the original series. This transforms the original monotone series into an alternating series. In the second step, the convergence of the transformed series is accelerated with the help of suitable nonlinear sequence transformations that are known to be particularly powerful for alternating series. Some theoretical aspects of our approach are discussed. The efficiency, numerical stability, and wide applicability of the combined nonlinear-condensation transformation is illustrated by a number of examples. We discuss the evaluation of special functions close to or on the boundary of the circle of convergence, even in the vicinity of singularities. We also consider a series of products of spherical Bessel functions, which serves as a model for partial wave expansions occurring in quantum electrodynamic bound state calculations.

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I. INTRODUCTION

Divergent and slowly convergent series occur abundantly in the mathematical and physical sciences. Accordingly, there is an extensive literature on numerical techniques which convert a divergent or slowly convergent series into a new series with hopefully better numerical properties. An overview of the existing sequence transformations as well as many references can be found in books by Wimp [1] and Brezinski and Redivo Zaglia [2]. The historical development of these techniques up to 1945 is described in a monograph by Brezinski [3], and the more recent developments are discussed in an article by Brezinski [4].

A very important class of sequences $\{s_n\}_{n=0}^{\infty}$ is characterized by the asymptotic condition

$$\lim_{n\to\infty} \frac{s_{n+1} - s}{s_n - s} \;=\; \rho \,, \tag{1.1}$$

which closely resembles the well known ratio test for infinite series. Here, $s = s_\infty$ is the limit of this sequence as $n \to \infty$. A convergent sequence satisfying condition (1.1) with $|\rho| < 1$ is called *linearly* convergent, and it is called *logarithmically* convergent if $\rho = 1$.

If the elements of the sequence $\{s_n\}_{n=0}^{\infty}$ in Eq. (1.1) are the partial sums $s_n = \sum_{k=0}^{n} a_k$ of an infinite series, and if ρ satisfies either $\rho = 1$ or $0 < \rho < 1$ (these are the only cases which will be considered in this article), then there exists an integer $N \geq 0$ such that all terms a_k with $k \geq N$ have the same sign. Hence, series of this kind can be split up into a finite sum containing the leading terms with the irregular signs, and a monotone series whose terms all have the same sign. The subject of this article is the efficient and reliable evaluation of series that exhibit these properties.

The partial sums of a nonterminating Gaussian hypergeometric series

$$
{}_2F_1(a,b;c;z) \;=\; \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}\,, \tag{1.2}
$$

where $(a)_m = \Gamma(a+m)/\Gamma(a) = a(a+1)\ldots(a+m-1)$ is a Pochhammer symbol, or its generalization

$$
\begin{aligned}
&{}_{p+1}F_p\bigl(\alpha_1,\ldots,\alpha_{p+1};\beta_1,\ldots,\beta_p;z\bigr) \\
&\quad = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots (\alpha_{p+1})_m}{(\beta_1)_m \ldots (\beta_p)_m} \frac{z^m}{m!}\,,
\end{aligned} \tag{1.3}
$$

which converge for $|z| < 1$ and diverge for $|z| > 1$, are typical examples of linearly convergent sequences with $\rho = z$.

The partial sums of the Dirichlet series

$$
\zeta(z) \;=\; \sum_{m=0}^{\infty} (m+1)^{-z} \tag{1.4}
$$

for the Riemann zeta function converge logarithmically if $\mathrm{Re}(z) > 1$. This follows from the following asymptotic estimate ($n \to \infty$) of the truncation error (p. 21 of Ref. [1]):

$$
\begin{aligned}
&\zeta(z) \;-\; \sum_{m=0}^{n} (m+1)^{-z} \\
&\quad = \frac{(n+1)^{1-z}}{z-1} \;-\; \frac{1}{2(n+1)^z} \;+\; O\bigl(n^{-z-1}\bigr)\,.
\end{aligned} \tag{1.5}
$$

The Dirichlet series (1.4) is notorious for its extremely slow convergence if $\mathrm{Re}(z)$ is only slightly larger than one. In this case, the series can only be used for the computation of $\zeta(z)$ if it is combined with suitable convergence acceleration methods like the Euler-Maclaurin summation (see for instance Chapter 8 of Ref. [5], p. 379 of Ref. [6], or Chapter 6 of Ref. [7]).

The acceleration of linear convergence is comparatively simple, both theoretically and practically, as long as ρ in Eq. (1.1) is not too close to one. With the help of Germain-Bonne's formal theory of convergence acceleration [8] and its extension (Section 12 of Ref. [9]), it can be decided whether a sequence transformation is capable of accelerating linear convergence or not. Moreover, many sequence transformations are known which are capable of accelerating linear convergence *effectively*. Examples are the Δ^2 process, which is usually attributed to Aitken [10] although it is in fact much older (p. 90 of Ref. [3]), Wynn's epsilon algorithm [11], which produces Padé approximants if the input data are the partial sums of a formal power series, or Levin's sequence transformation [12] and generalizations (Sections 7 - 9 of Ref. [9]), which require as input data not only the elements of the sequence to be transformed but also explicit estimates for the corresponding truncation errors.

The acceleration of logarithmic convergence is much more difficult than the acceleration of linear convergence. Delahaye and Germain-Bonne [13] showed that no sequence transformation can exist which is able to accelerate the convergence of *all* logarithmically convergent sequences. Consequently, in the case of logarithmic convergence the success of a convergence acceleration process cannot be guaranteed unless additional information is available. Also, an analogue of Germain-Bonne's formal theory of linear convergence acceleration [8] cannot exist.

In spite of these complications, many sequence transformations are known which work reasonably well at least for suitably restricted subsets of the class of logarithmically convergent sequences. Examples are Richardson extrapolation [14], Wynn's rho algorithm [15] and its iteration (Section 6 of Ref. [9]) as well as Osada's modification of the rho algorithm [16], Brezinski's theta algorithm [17] and its iteration (Section 10 of Ref. [9]), Levin's u and v transformations [12] and related transformations (Sections 7 - 9 of Ref. [9]), and the modification of the Δ^2 process by Bjørstad, Dahlquist, and Grosse [18]. However, there is a considerable amount of theoretical and empirical evidence that sequence transformations are in general less effective in the case of logarithmic convergence than in the case of linear convergence.

Numerical stability is a very important issue. A sequence transformation can only accelerate convergence if it succeeds in extracting some additional information about the index-dependence of the truncation errors from a finite set s_n, s_{n+1}, …, s_{n+k} of input data. Normally, this is done by forming arithmetic expressions involving higher weighted differences. However, forming higher weighted differences is a potentially unstable process which can easily lead to a serious loss of significant digits or even to completely nonsensical results.

If the input data are the partial sums of a *strictly alternating* series, the formation of higher weighted differences is normally a remarkably stable process. Hence, a serious loss of significant digits is not to be expected if the partial sums of a strictly alternating series are used as input data in a convergence acceleration or summation process. If, however, the input data are the partial sums of a *monotone* series, numerical instabilities due to cancellation are quite likely, in particular if convergence is very slow. Thus, if the sequence to be transformed either converges linearly with a value of ρ in Eq. (1.1) that is only slightly smaller than one, or if it converges logarithmically ($\rho = 1$), numerical instabilities are a serious problem and at least some loss of significant digits is to be expected.

Generally, the sequence transformations mentioned above are not able to determine the limit of a logarithmically convergent series, whose terms ultimately all have the same sign, with an accuracy close to machine accuracy. This restricts the practical usefulness of sequence transformations severely, e.g., in FORTRAN calculations with a fixed precision.

In this paper, we show that these stability problems can often be overcome by transforming slowly convergent

monotone series not by straightforward application of a *single* sequence transformation but by a combination of two *different* transformations.

In the *first* step, a monotone series is transformed into a strictly alternating series with the help of a *condensation transformation*. This transformation was first mentioned on p. 126 of Ref. [19] and only later published by Van Wijngaarden [20]. Later, the Van Wijngaarden transformation was studied by Daniel [21], and recently it was rederived by Pelzl and King [22], who used it for the high-precision calculation of atomic three-electron interaction integrals of explicitly correlated wave functions.

In the *second* step, the convergence of the resulting alternating series is accelerated by suitable *nonlinear sequence transformations* [9,12] which are known to be very powerful in the case of alternating series. Since the transformation of alternating series is a remarkably stable process, the limits of even extremely slowly convergent monotone series can be determined with an accuracy close to machine accuracy. Conceptually, but not technically our approach resembles that of Brezinski, Delahaye, and Germain-Bonne [23] who proposed to extract a linearly convergent subsequence from a logarithmically convergent input sequence by a selection process.

In this article, we will call our approach, which consists of the Van Wijngaarden *condensation* transformation and the subsequent *nonlinear* sequence transformation, the *combined nonlinear-condensation transformation* (CNCT).

The CNCT is not restricted to logarithmically convergent series. It can also be used in the case of a linearly convergent monotone series with a value of ρ in Eq. (1.1) close to one. Typically, this corresponds to a power series whose coefficients ultimately all have the same sign and whose argument is positive and close to the boundary of the circle of convergence. We will present some examples which show that the CNCT works even if the argument of the power series is very close to a singularity.

Our approach requires the evaluation of terms of the original monotone series with high indices. Consequently, our two-step approach is computationally more demanding than the application of a single sequence transformation, and it cannot be applied if only a few terms of a slowly convergent series are available. In spite of these restrictions, we believe that the CNCT is very useful at least for certain problems since it is able to produce highly accurate results that can only be accomplished otherwise with a considerably greater numerical effort.

Special functions are defined and, in many cases, also evaluated via series expansions. The evaluation of special functions is an old problem of numerical mathematics with a very extensive literature (compare for example the books by Luke [24,25], Van Der Laan and Temme [26], and Zhang and Jin [27]). Nevertheless, there is still a considerable amount of research going on and many new algorithms for the computation of special functions have been developed recently (compare for example the papers by Lozier and Olver [28] and Lozier [29] and the long lists of references therein). We believe that sequence transformations in general and the CNCT in particular are useful tools for the evaluation of special functions. Of course, this is also true for problems in theoretical and computational physics.

This paper is organized as follows: In Section II, we describe the Van Wijngaarden transformation. In Section III, we discuss the nonlinear sequence transformations which we use for the acceleration of the resulting alternating series. In Section IV, we apply the CNCT to the Riemann zeta function. In Section V, we consider the evaluation of the Lerch transcendent and related functions with arguments on or close to the boundary of the circle of convergence. In Section VI, we examine the evaluation of the generalized hypergeometric series ${}_{p+1}F_p$ ($p \geq 2$) with unit argument or with an argument z which is only slightly smaller than one. In Section VII, we discuss the evaluation of an infinite series involving Bessel and Hankel functions. In Appendix A, we discuss the efficiency of sequence transformations in the case of monotone series or strictly alternating series in more detail. Finally, in Appendix B we discuss exactness properties of the sequence transformations which we use in the second step for the acceleration of the convergence of the resulting alternating series.

The example of Section VII serves as a model problem for slowly convergent series, which occur in quantum electrodynamic bound state calculations and which were treated successfully with the methods discussed in this paper [30]. Therefore, we expect the CNCT to be a general computational tool for the evaluation of slowly convergent sums over intermediate angular momenta which arise from the decomposition of relativistic propagators in QED bound state calculations into partial waves.

All calculations were done in Mathematica[1] with a relative accuracy of 16 decimal digits [31]. In this way, we simulate the usual DOUBLE PRECISION accuracy in FORTRAN.

II. THE VAN WIJNGAARDEN TRANSFORMATION

Let us assume that the partial sums

$$\sigma_n \;=\; \sum_{k=0}^{n} a(k) \tag{2.1}$$

[1] Certain commercial equipment, instruments, or materials are identified in this paper to foster understanding. Such identification does not imply recommendation or endorsement by the National Institute of Standards and Technology, nor does it imply that the materials or equipment identified are necessarily the best available for the purpose.

of an infinite series converge either linearly or logarithmically to some limit $\sigma = \sigma_\infty$ as $n \to \infty$. We also assume that all terms $a(k)$ have the same sign, i.e., the series $\sum_{k=0}^{\infty} a(k)$ is a monotone series.

Following Van Wijngaarden [20], we transform the original series into an alternating series $\sum_{j=0}^{\infty} (-1)^j \mathbf{A}_j$ according to

$$\sum_{k=0}^{\infty} a(k) = \sum_{j=0}^{\infty} (-1)^j \mathbf{A}_j \, , \tag{2.2}$$

$$\mathbf{A}_j = \sum_{k=0}^{\infty} \mathbf{b}_k^{(j)} \, , \tag{2.3}$$

$$\mathbf{b}_k^{(j)} = 2^k \, a(2^k \, (j+1) - 1) \, . \tag{2.4}$$

Obviously, the terms $\mathbf{A}_j$ defined in Eq. (2.3) all have the same sign if the terms $a(k)$ of the original series all have the same sign. In the sequel, the quantities $\mathbf{A}_j$ will be referred to as the *condensed series*, and the series $\sum_{j=0}^{\infty} (-1)^j \mathbf{A}_j$ will be referred to as the *transformed alternating series*, or alternatively as the *Van Wijngaarden transformed series*.

We call the Van Wijngaarden transformation a condensation transformation because its close connection to Cauchy's condensation theorem (p. 28 of Ref. [32] or p. 121 of Ref. [33]). Given a monotone series $\sum_{k=0}^{\infty} a(k)$ with terms that satisfy $|a(k+1)| < |a(k)|$, Cauchy's condensation theorem states that $\sum_{k=0}^{\infty} a(k)$ converges if and only if the first condensed series $\mathbf{A}_0$ defined in Eq. (2.3) converges.

In Eq. (2.3), the indices of the terms of the original series are chosen in such a way that sampling at very high indices takes place (according to Eq. (2.4), the indices of the terms of the original series grow exponentially). In this way, we obtain information about the behaviour of the terms of the original series at high indices.

Moreover, if the terms of the original series behave asymptotically ($n \to \infty$) either like $a(n) \sim n^{-1-\epsilon}$ with $\epsilon > 0$ or like $a(n) \sim n^\beta r^n$ with $0 < r < 1$ and β real, then the terms of the original series become negligibly small after a few evaluations. Specifically, the series (2.3) for the terms $\mathbf{A}_j$ converges linearly in these cases.

When summing over k in Eq. (2.3), we found that it is normally sufficient to terminate this sum when the last term is smaller than the desired accuracy. Typically, 20 to 30 terms are needed for a relative accuracy of 10^{-14} in the final result for the condensed sum.

It should be possible to accelerate the convergence of the series (2.3) for $\mathbf{A}_j$. Since, however, this is a monotone series, it is not clear whether and how many digits would be lost in the convergence acceleration process. Consequently, we prefer to perform a safe and straightforward evaluation of the condensed sums $\mathbf{A}_j$ and add up the terms of the series until convergence is reached, although this is most likely not the most efficient approach.

The transformation from a monotone series to a strictly alternating series according to Eqs. (2.2) - (2.4) is essentially a reordering of the terms $a(k)$ of the original series. This is seen as follows. We first define the partial sums

$$\mathbf{S}_n \; = \; \sum_{j=0}^{n} (-1)^j \, \mathbf{A}_j \tag{2.5}$$

of the Van Wijngaarden transformed original series. It can be shown easily that $\mathbf{S}_n$ with $n \geq 0$ reproduces the partial sum σ_n, Eq. (2.1), which contains the first $n+1$ terms of the original series. To illustrate this procedure, we present Table I. Formal proofs of the correctness of this rearrangement can be found in Ref. [21] or in the Appendix of Ref. [22].

Insert Table I here

Daniel [21] was able to formulate some mild conditions which guarantee that the limits σ and $\mathbf{S}$ of the partial sums (2.1) and (2.5), respectively, simultaneously exist and are equal:

$$\lim_{n \to \infty} \sigma_n \; = \; \sigma \; = \; \lim_{n \to \infty} \mathbf{S}_n \; = \; \mathbf{S} \, . \tag{2.6}$$

For example, in the Corollary on p. 92 of Ref. [21] it was shown that if a strictly decreasing sequence $\{M_k\}_{k=0}^{\infty}$ of positive bounds exists which satisfy $|a(k)| \leq M_k$ for all $k \geq 0$ *and* if $\sum_{k=0}^{\infty} M_k < \infty$ holds, then the original monotone series and the Van Wijngaarden transformed series on the right-hand side of Eq. (2.2) both converge to the same limit (i.e., $\sigma = \mathbf{S}$ holds). This useful criterion is fulfilled by all series considered in this paper.

III. NONLINEAR SEQUENCE TRANSFORMATIONS

The series (2.3) for the terms of the Van Wijngaarden transformed series can be rewritten as follows:

$$\mathbf{A}_j \; = \; a(j) \; + \; 2a(2j+1) \; + \; 4a(4j+3) \; + \; \ldots \, . \tag{3.1}$$

Since the terms $a(k)$ of the original series have by assumption the same sign, we immediately observe

$$|\mathbf{A}_j| \geq |a_j| \, . \tag{3.2}$$

Consequently, the Van Wijngaarden transformation, Eqs. (2.2) - (2.4), does not lead to an alternating series whose terms decay more rapidly in magnitude than the terms of the original monotone series. Thus, an acceleration of convergence can only be achieved if the partial sums (2.5) of the Van Wijngaarden transformed series are used as input data in a convergence acceleration process.

Since the Van Wijngaarden transformed series on the right-hand side of Eq. (2.2) is alternating if the terms of the original series all have the same sign, it is recommended to choose a suitable convergence acceleration

method which is particularly powerful in the case of alternating series. A judicious choice of the convergence accelerator is of utmost importance since it ultimately decides whether our approach is numerically useful or not.

Daniel [21] used the Euler transformation (Eq. (3.6.27) on p. 16 of Ref. [34])):

$$\sum_{k=0}^{\infty} (-1)^k u_k \; = \; \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} \, \Delta^k u_0 \, . \tag{3.3}$$

Here, Δ is the (forward) difference operator defined by $\Delta f(n) = f(n+1) - f(n)$, and

$$\Delta^k u_0 \; = \; (-1)^k \sum_{m=0}^{k} (-1)^m \binom{k}{m} u_m \, . \tag{3.4}$$

The Euler transformation, which was published in its original version in 1755 on p. 281 of Ref. [35], is a series transformation which is specially designed for alternating series. It is treated in many books on numerical mathematics. Nevertheless, the Euler transformation is not a particularly efficient accelerator for the Van Wijngaarden transformed series considered in this article. In Table II, we show some explicit results obtained by the Euler transformation. We also applied the Euler transformation to all other Van Wijngaarden transformed series presented in this paper, and we consistently observed that it is clearly less powerful than the transformations we used.

Much better results can be expected from the more modern nonlinear sequence transformations which transform a sequence $\{s_n\}_{n=0}^{\infty}$, whose elements may be the partial sums of an infinite series, into a new sequence $\{s'_n\}_{n=0}^{\infty}$ with better numerical properties [1,2,9].

The basic assumption of a sequence transformation is that the elements of the sequence $\{s_n\}_{n=0}^{\infty}$ to be transformed can be partitioned into a limit s and a remainder r_n according to

$$s_n \; = \; s \, + \, r_n \tag{3.5}$$

for all $n \ge 0$. Only in the case of some more or less trivial model cases will a sequence transformation be able to determine the limit s of the sequence $\{s_n\}_{n=0}^{\infty}$ exactly after a *finite* number of steps. Hence, the elements of the transformed sequence $\{s'_n\}_{n=0}^{\infty}$ can be partitioned into the same limit s and a transformed remainder r'_n according to

$$s'_n \; = \; s \, + \, r'_n \tag{3.6}$$

for all $n \ge 0$. In general, the transformed remainders r'_n are nonzero for all finite values of n. However, convergence is obviously increased if the transformed remainders $\{r'_n\}_{n=0}^{\infty}$ vanish more rapidly than the original remainders $\{r_n\}_{n=0}^{\infty}$:

$$\lim_{n\to\infty} \frac{r_{n+1}}{r_n} \; = \; \lim_{n\to\infty} \frac{s_{n+1} - s}{s_n - s} \; = \; 0 \, . \tag{3.7}$$

The best known nonlinear sequence transformation is probably Wynn's epsilon algorithm [11] which produces Padé approximants if the input data are the partial sums of a formal power series. The epsilon algorithm is also known to work well in the case of alternating series. Consequently, it is an obvious idea to use the epsilon algorithm for the acceleration of the alternating series which we obtain via the Van Wijngaarden transformation.

However, much better results can be obtained by applying sequence transformations which use explicit remainder estimates as input data in addition to the partial sums of the series to be transformed. Consequently, in this article we use exclusively the following two sequence transformations by Levin [12] and Weniger [9], respectively:

$$\begin{aligned}
\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n) \; &= \; \frac{\Delta^k \left\{ (n+\beta)^{k-1} \, s_n/\omega_n \right\}}{\Delta^k \left\{ (n+\beta)^{k-1}/\omega_n \right\}} \\
&= \frac{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \, \frac{s_{n+j}}{\omega_{n+j}}}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \, \frac{1}{\omega_{n+j}}} \, ,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n) \; &= \; \frac{\Delta^k \left\{ (n+\beta)_{k-1} \, s_n/\omega_n \right\}}{\Delta^k \left\{ (n+\beta)_{k-1}/\omega_n \right\}} \\
&= \frac{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)_{k-1}}{(\beta+n+k)_{k-1}} \, \frac{s_{n+j}}{\omega_{n+j}}}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)_{k-1}}{(\beta+n+k)_{k-1}} \, \frac{1}{\omega_{n+j}}} \, .
\end{aligned} \tag{3.9}$$

Here, $\{s_n\}_{n=0}^{\infty}$ is the sequence to be transformed, and $\{\omega_n\}_{n=0}^{\infty}$ is a sequence of truncation error estimates. The shift parameter β has to be positive in order to permit $n = 0$ in Eqs. (3.8) and (3.9). The most obvious choice is $\beta = 1$. This has been the choice in virtually all previous applications of these sequence transformations, and we will use $\beta = 1$ unless explicitly stated. However, we show in Appendix B that in some cases it may be very advantageous to choose other values for β.

The numerator and denominator sums in Eqs. (3.8) and (3.9) can also be computed with the help of the three-term recursions (Eq. (7.2-8) of Ref. [9])

$$\begin{aligned}
L_{k+1}^{(n)}(\beta) \; &= \; L_k^{(n+1)}(\beta) \\
&- \frac{(\beta+n+k)(\beta+n+k)^{k-1}}{(\beta+n+k+1)^k} \, L_k^{(n)}(\beta)
\end{aligned} \tag{3.10}$$

and (Eq. (8.3-7) of Ref. [9])

$$\begin{aligned}
S_{k+1}^{(n)}(\beta) \; &= \; S_k^{(n+1)}(\beta) \\
&- \frac{(\beta+n+k)(\beta+n+k-1)}{(\beta+n+2k)(\beta+n+2k-1)} \, S_k^{(n)}(\beta) \, .
\end{aligned} \tag{3.11}$$

The initial values $L_0^{(n)}(\beta) = S_0^{(n)}(\beta) = s_n/\omega_n$ and $L_0^{(n)}(\beta) = S_0^{(n)}(\beta) = 1/\omega_n$ produce the numerator and denominator sums, respectively, of $\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n)$ and $\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n)$.

The performance of the sequence transformations $\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n)$ and $\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n)$ depends crucially on the remainder estimates $\{\omega_n\}_{n=0}^{\infty}$. Under exceptionally favourable circumstances it may be possible to construct explicit approximations to the truncation errors of the input sequence $\{s_n\}_{n=0}^{\infty}$. In most cases of practical interest this is not possible, since only the numerical values of a finite number of sequence elements is available. Hence, in practice one has to determine the remainder estimates from these numerical values.

On the basis of purely heuristic and asymptotic arguments, Levin [12] proposed some simple remainder estimates which – when used in Eq. (3.8) – lead to Levin's u, t, and v transformations. These remainder estimates can also be used in Eq. (3.9) (Section 8.4 of Ref. [9]).

However, in the case of a strictly alternating series the best *simple* truncation error estimate is the first term neglected in the partial sum (p. 259 of Ref. [33]). This is also the best simple estimate for the truncation error of a strictly alternating nonterminating hypergeometric series ${}_2F_0(\alpha, \beta; -z)$ with $\alpha, \beta, z > 0$, which diverges strongly for every $z \neq 0$ (Theorem 5.12-5 of Ref. [36]). Consequently, in the case of convergent or divergent alternating series it is a natural idea to use the first term neglected in the partial sum as the remainder estimate, as proposed by Smith and Ford [37].

In this article, we always accelerate the convergence of the partial sums $\mathbf{S}_n$ of the Van Wijngaarden transformed series (2.3) which is strictly alternating if the original series is monotone. Thus, we use exclusively the sequence transformations (3.8) and (3.9) in combination with the remainder estimate proposed by Smith and Ford [37]:

$$\omega_n \;=\; \Delta \mathbf{S}_n \;=\; (-1)^{n+1} \mathbf{A}_{n+1} \, . \tag{3.12}$$

This yields the following sequence transformations (Eqs. (7.3-9) and (8.4-4) of Ref. [9]):

$$\begin{aligned} d_k^{(n)}\big(\beta, \mathbf{S}_n\big) \;&=\; \mathcal{L}_k^{(n)}(\beta, \mathbf{S}_n, \Delta \mathbf{S}_n) \\ &= \mathcal{L}_k^{(n)}(\beta, \mathbf{S}_n, (-1)^{n+1} \mathbf{A}_{n+1}) \, , \end{aligned} \tag{3.13}$$

$$\begin{aligned} \delta_k^{(n)}\big(\beta, \mathbf{S}_n\big) \;&=\; \mathcal{S}_k^{(n)}(\beta, \mathbf{S}_n, \Delta \mathbf{S}_n) \\ &= \mathcal{S}_k^{(n)}(\beta, \mathbf{S}_n, (-1)^{n+1} \mathbf{A}_{n+1}) \, . \end{aligned} \tag{3.14}$$

Unless explicitly stated, we use $\beta = 1$. In the applications described in the following, it will be clear from the context which monotone series was transformed according to Van Wijngaarden such as to produce the input data for the nonlinear sequence transformations which are the partial sums $\mathbf{S}_n$ of the alternating series (2.3).

Alternative remainder estimates for the sequence transformations (3.8) and (3.9) were discussed in Sections 7 and 8 of Ref. [9] or in Refs. [38,39].

¿From a purely theoretical point of view, the sequence transformations $d_k^{(n)}$ and $\delta_k^{(n)}$ as well as their parent transformations $\mathcal{L}_k^{(n)}$ and $\mathcal{S}_k^{(n)}$ have the disadvantage that no general convergence proof is known. Only for some special model problems could rigorous convergence proofs be obtained (Refs. [40–42] or Sections 13 and 14 of Ref. [9]). However, there is overwhelming empirical evidence that $d_k^{(n)}$ and $\delta_k^{(n)}$ work very well in the case of convergent or divergent alternating series for instance as they occur in special function theory [2,9,12,37,43–48] or in quantum mechanical perturbation theory [44,45,49–55].

Pelzl and King [22] only used the Levin transformation $d_k^{(n)}\big(\beta, \mathbf{S}_n\big)$ for the acceleration of the convergence of Van Wijngaarden transformed alternating series. However, we shall show that the closely related transformation $\delta_k^{(n)}\big(\beta, \mathbf{S}_n\big)$ frequently gives better results.

Application of the sequence transformations (3.13) and (3.14) to the partial sums $\mathbf{S}_n$ of the infinite series (2.3) produces doubly indexed sets of transforms, which depend on the starting index n of the input data and the transformation order k:

$$\{\mathbf{S}_n, \mathbf{S}_{n+1}, \ldots, \mathbf{S}_{n+k}, \mathbf{S}_{n+k+1}\} \to T_k^{(n)} \, . \tag{3.15}$$

Here, $T_k^{(n)}$ stands for either $d_k^{(n)}\big(\beta, \mathbf{S}_n\big)$ or $\delta_k^{(n)}\big(\beta, \mathbf{S}_n\big)$.

The transforms can be displayed in a rectangular scheme called the *table* of the transformation. In principle, there is an unlimited variety of different paths, on which we can move through the table in a convergence acceleration or summation process. In this article, we always proceed on such a path that for a given set of input data, we use the transforms with the highest possible transformation order as approximations to the limit of the series to be transformed. Thus, given the input data $\{\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_m, \mathbf{S}_{m+1}\}$ we always use the sequence $\left\{T_0^{(0)}, T_1^{(0)}, \ldots, T_m^{(0)}\right\}$ of transforms as approximations to the limit (compare for instance Eqs. (7-5.4) and (7-5.5) of Ref. [9]), where $T_m^{(0)}$ should provide the best approximation since it has the highest transformation order.

In Appendix A, we explain why the transformations $\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n)$ and $\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n)$ are much more effective in the case of an alternating series than in the case of a monotone series.

IV. THE RIEMANN ZETA FUNCTION

The Riemann zeta function is discussed in most books on special functions. Particularly detailed treatments can be found in monographs by Edwards [7], Titchmarsh [56], Ivić [57], and Patterson [58]. Many applications of the zeta function in theoretical physics are described in books by Elizalde, Odintsov, Romeo, Bytsenko, and Zerbini [59] and by Elizalde [60].

In this section, we discuss how the CNCT can be used for the evaluation of the Riemann zeta function, start-

ing from the logarithmically convergent Dirichlet series (1.4). We do not claim that our approach is necessarily more powerful than other, more specialized techniques for the evaluation of the Riemann zeta function. However, because of its simplicity, the Dirichlet series (1.4) is particularly well suited for an illustration of our approach.

In the case of the Riemann zeta function, the terms $\mathbf{b}_k^{(j)}$ of the series (2.3) for $\mathbf{A}_j$ are given by

$$\mathbf{b}_k^{(j)} \;=\; \frac{\left(2^{1-z}\right)^k}{(j+1)^z}\,. \tag{4.1}$$

Thus, the series (2.3) is a power series in 2^{1-z} which converges linearly. Moreover, it is essentially the geometric series in the variable 2^{1-z} that can be expressed in closed form according to

$$\mathbf{A}_j \;=\; \frac{1}{1-2^{1-z}}\,\frac{1}{(j+1)^z}\,. \tag{4.2}$$

Inserting this into the infinite series on the right-hand side of Eq. (2.2) yields the following transformed alternating series for the Riemann zeta function:

$$\zeta(z) \;=\; \frac{1}{1-2^{1-z}}\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)^z}\,. \tag{4.3}$$

An alternative proof of the validity of this series expansion can be found in Section 2.2 of Titchmarsh's book [56].

The terms of the alternating series (4.3) decay as slowly in magnitude as the terms of the logarithmically convergent Dirichlet series (1.4). Nevertheless, the series (4.3) offers some substantial computational advantages because it is alternating. For example, it converges for $\mathrm{Re}(z) > 0$, whereas the Dirichlet series converges only for $\mathrm{Re}(z) > 1$. For $\mathrm{Re}(z) \le 0$ both the Dirichlet series (1.4) and the alternating series (4.3) diverge. However, the alternating series can provide an analytic continuation of the Riemann zeta function for $\mathrm{Re}(z) \le 0$ if it is combined with a suitable summation process. For this purpose, we can use the same nonlinear transformations (3.13) and (3.14) as in the case of a *convergent* Van Wijngaarden transformed alternating series.

This observation extends the applicability of the CNCT, which was originally designed for convergent monotone series only. Of course, this extension to divergent series is only possible if the monotone series (2.3) for $\mathbf{A}_j$ can be summed. Normally, the summation of a monotone divergent series is very difficult. However, in the case of the Riemann zeta function this is trivial since the series (2.3) is – as remarked above – essentially the geometric series in the variable 2^{1-z} which can be expressed in closed form according to Eq. (4.2), no matter whether $|2^{1-z}| < 1$ or $|2^{1-z}| > 1$ holds.

Our first numerical example is the zeta function with argument $z = 1.01$. The Dirichlet series (1.4) converges for this argument, but its convergence is so slow that it is computationally useless. It follows from the truncation error estimate (1.5) that on the order of 10^{600} terms would be needed to achieve the modest relative accuracy of 10^{-6} if the Dirichlet series were summed by adding up the terms. A much more efficient evaluation of $\zeta(z)$ near $\mathrm{Re}(z) = 1$ is based on the Euler-Maclaurin formula (Eqs. (2.01) and (2.02) on p. 285 of Ref. [5])

$$\sum_{n=N}^{M} f(n) \;=\; \int_N^M f(x)\,\mathrm{d}x \;+\; \frac{1}{2}\left[f(N)+f(M)\right] \;+\; \sum_{j=1}^{q}\frac{B_{2j}}{(2j)!}\left[f^{(2j-1)}(M) - f^{(2j-1)}(N)\right] \;+\; R_q\,, \tag{4.4a}$$

$$R_q \;=\; -\frac{1}{(2q)!}\int_N^M B_{2q}\bigl(x - [\![x]\!]\bigr)\, f^{(2q)}(x)\,\mathrm{d}x\,. \tag{4.4b}$$

Here, $[\![x]\!]$ is the integral part of x, i.e., the largest integer m satisfying $m \le x$, $B_k(x)$ is a Bernoulli polynomial defined by the generating function (Eq. (1.06) on p. 281 of Ref. [5])

$$\frac{t\exp(xt)}{\exp(t)-1} \;=\; \sum_{m=0}^{\infty} B_m(x)\,\frac{t^m}{m!}\,, \qquad |t| < 2\pi\,, \tag{4.5}$$

and

$$B_m \;=\; B_m(0) \tag{4.6}$$

is a Bernoulli number (p. 281 of Ref. [5]).

In the next step, we rewrite the Dirichlet series (1.4) as follows,

$$\zeta(z) \;=\; \sum_{m=0}^{N-1}\frac{1}{(m+1)^z} \;+\; \sum_{m=N}^{\infty}\frac{1}{(m+1)^z}\,, \tag{4.7}$$

and apply the Euler-Maclaurin formula (4.4) to the infinite series on the right-hand side. With $N = 100$, we easily obtain the value of the zeta function accurate to more than 15 significant decimal digits. The value of $\zeta(1.01)$ to 15 decimal places is

$$10^{-3}\,\zeta(1.01) \;=\; 0.100\,577\,943\,338\,497\,. \tag{4.8}$$

We present this result (as well as all other subsequent numerical results) normalized to a number in the interval $(0, 1)$ by a multiplicative power of 10.

In Table II, we apply the Euler transformation (3.3) and the nonlinear sequence transformations (3.13) and (3.14) to the partial sums of the alternating series (4.3) with $z = 1.01$. We list as a function of n the partial sums

$$\mathbf{S}_n \;=\; \frac{1}{1-2^{1-z}}\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)^z} \tag{4.9}$$

of the alternating series (4.3), the partial sums

$$\mathbf{E}_n \; = \; \frac{1}{1-2^{1-z}} \sum_{k=0}^{n} \frac{1}{2^{k+1}} \sum_{m=0}^{k} \frac{(-1)^m \binom{k}{m}}{(m+1)^z} \qquad (4.10)$$

of the Euler transformed series (3.3), and the nonlinear sequence transformations $d_n^{(0)}\bigl(1, \mathbf{S}_0\bigr)$ and $\delta_n^{(0)}\bigl(1, \mathbf{S}_0\bigr)$.

Insert Table II here

The results in Table II show that the Euler transformation is in the case of the alternating series (4.3) much less effective than the nonlinear sequence transformations $d_n^{(0)}\bigl(1, \mathbf{S}_0\bigr)$ and $\delta_n^{(0)}\bigl(1, \mathbf{S}_0\bigr)$. When we applied Wynn's epsilon algorithm [11] to the alternating series (4.3), we also obtained clearly inferior results. This is in agreement with the observations of Pelzl and King [22].

Thus, in the following we will only consider the nonlinear transformations (3.13) and (3.14) which use explicit remainder estimates.

In Table II, we present apparently redundant data since our transformation results in the last two columns do not change for $n \geq 12$. However, we include these results here as well as below in order to demonstrate that our transformation remains stable even if we increase the transformation order.

If the argument z is zero or a negative integer, the Riemann zeta function satisfies (p. 807 of Ref. [34]):

$$\zeta(0) = -\frac{1}{2}\,, \qquad \zeta(-2m) \; = \; 0\,, \qquad (4.11a)$$

$$\zeta(1-2m) = -\frac{B_{2m}}{2m}\,, \qquad m \; = \; 1, 2, \ldots\,. \qquad (4.11b)$$

Here, B_{2m} is a Bernoulli number defined in Eq. (4.6).

Our second numerical example is the zeta function with argument $z = -1$ which, because $B_2 = 1/6$, satisfies

$$\zeta(-1) \; = \; -1/12 \; = \; -0.0833\,333\ldots\,. \qquad (4.12)$$

As remarked above, the alternating series (4.3) diverges for $z = -1$. However, our results in Table III show that this series can be summed effectively by the nonlinear sequence transformations (3.13) and (3.14).

Insert Table III here

The results in Table III indicate that the sequence transformation $\delta_n^{(0)}\bigl(1, \mathbf{S}_0\bigr)$ is exact for the alternating series (4.3) with $z = -1$. The exactness of the sequence transformations $\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n)$, Eq. (3.8), and $\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n)$, Eq. (3.9), for the infinite series (4.3) and for related series is discussed in Appendix B.

There is considerable research going on in connection with the Riemann-Siegel conjecture [61–64]. In this context, it is necessary to evaluate the Riemann zeta function on the so-called critical line $z = 1/2 + t\ i$ ($-\infty < t < \infty$). This can also be accomplished by applying the sequence transformations (3.13) and (3.14) to the partial sums of the alternating series (4.3). In Table IV, where we treat the case $z = 1/2 + 13.7\ i$, the Levin transformation outperforms the Weniger transformation. For limitations of space, we do not present the partial sums of the transformed alternating series in Table IV.

Insert Table IV here

V. THE LERCH TRANSCENDENT AND RELATED FUNCTIONS

The Lerch transcendent (p. 32 of Ref. [65])

$$\Phi(z, s, \alpha) \; = \; \sum_{n=0}^{\infty} \frac{z^n}{(\alpha+n)^s}\,, \qquad |z| < 1\,, \qquad (5.1)$$

contains many special functions as special cases, for example the Riemann zeta function

$$\zeta(s) \; = \; \Phi(1, s, 1) \qquad (5.2)$$

or the Jonquière function (p. 33 of Ref. [65])

$$F(z, s) \; = \; \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^s} \; = \; z\,\Phi(z, s, 1)\,. \qquad (5.3)$$

In the physics literature, the Jonquière function is usually called generalized logarithm or polylogarithm, and the following notation is used:

$$\mathrm{Li}_s(z) \; = \; \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)^s} \; = \; F(z, s)\,. \qquad (5.4)$$

This is also the notation and terminology used in the book by Lewin [66].

Lerch transcendents and their special cases, the polylogarithms, are ubiquitous in theoretical physics. They play a role in Bose-Einstein condensation [67,68], and they are particularly important in quantum field theory. For example, they occur in integrands of one-dimensional numerical integrations in quantum electrodynamic bound state calculations [69–73].

The series expansion (5.1) for $\Phi(z, s, \alpha)$ converges very slowly if the argument z is only slightly smaller than one. However, the CNCT can be used for an efficient and reliable evaluation of the Lerch transcendent and its special cases.

In our first example (Table V), we evaluate

$$\begin{aligned} 10^{-2}\,\mathrm{Li}_1(0.99999) &= -10^{-2}\,\ln(0.00001) \\ &= 0.115\,129\,254\,649\,702\,. \end{aligned} \qquad (5.5)$$

The series (5.4) for $\mathrm{Li}_s(z)$ converges linearly for $|z| < 1$, but

$$\mathrm{Li}_1(z) \; = \; \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} \; = \; -\ln(1-z) \qquad (5.6)$$

has a singularity at $z = 1$. Thus, in Table V we evaluate $\mathrm{Li}_1(z)$ in the immediate vicinity of a singularity.

Insert Table V here

In contrast, the series expansions

$$\mathrm{Li}_2(z) \; = \; \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)^2} \qquad (5.7)$$

and

$$\mathrm{Li}_3(z) \; = \; \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)^3} \qquad (5.8)$$

converge logarithmically for $z = 1$. In Tables VI and VII, we consider

$$10^{-1}\,\mathrm{Li}_2(0.99999) \; = \; 0.164\,480\,893\,699\,293\ldots \qquad (5.9)$$

and

$$10^{-1}\,\mathrm{Li}_3(0.99999) = 0.120\,204\,045\,438\,733\,, \qquad (5.10)$$

respectively. As in most previous examples, the Weniger transform $\delta_n^{(0)}\bigl(1, \mathbf{S}_0\bigr)$ produces in Tables VI and VII somewhat better results than the Levin transform $d_n^{(0)}\bigl(1, \mathbf{S}_0\bigr)$.

Insert Tables VI and VII here

In our last example of this section, we evaluate

$$10^4\,\Phi(0.99999, 2, 10000) \; = \; 0.798\,585\,139\,222\,548 \qquad (5.11)$$

with the help of the CNCT. The prefactor of 10000 is introduced so that the final result is of order one.

Insert Table VIII here

VI. THE GENERALIZED HYPERGEOMETRIC SERIES

The Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$, defined by the series expansion (1.2) for $|z| < 1$, is one of the most used and best understood special functions. It is discussed in virtually all books on special functions. A particularly detailed treatment can be found in the book by Slater [74].

The fact that the mathematical properties of the hypergeometric function ${}_2F_1$ are so well understood, greatly facilitates its computation. The series (1.2) does not suffice for the computation of a nonterminating ${}_2F_1$ since it converges only if $|z| < 1$. By contrast, the hypergeometric function ${}_2F_1$ is a multivalued function defined in the whole complex plane with branch points at $z = 1$ and ∞.

However, a hypergeometric function ${}_2F_1(a, b; c; z)$ can be transformed into the sum of two other ${}_2F_1$'s with arguments $w = 1-z$, $w = 1/z$, $w = 1/(1-z)$, or $w = 1-1/z$, respectively. Thus, the argument w of the two resulting hypergeometric series can normally be chosen in such a way that the two new series in w either converge, if the original series in z diverges, or they converge more rapidly if the original series converges too slowly to be numerically useful.

For example, if $|1 - z| < 1$ and if $c - a - b$ is not a positive or negative integer, then we can use the analytic continuation formula (Eq. (15.3.6) of Ref. [34])

$$\begin{aligned} & {}_2F_1(a, b; c; z) \\ & \quad = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1(a, b; a+b-c+1; 1-z) \\ & \qquad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\, (1-z)^{c-a-b} \\ & \qquad \times {}_2F_1(c-a, c-b; c-a-b+1; 1-z)\,. \end{aligned} \qquad (6.1)$$

Obviously, the two hypergeometric series in $1 - z$ will converge rapidly in the vicinity of $z = 1$. With the help of this or other analytic continuation formulas it is possible to compute a hypergeometric function ${}_2F_1(a, b; c; z)$ with arbitrary real argument $z \in (-\infty, +\infty)$ effectively via the resulting hypergeometric series (see p. 127 of Ref. [75] or Table I of Ref. [76]).

The situation is much less favourable in the case of the generalized hypergeometric series

$${}_3F_2\bigl(a, b, c; d, e; z\bigr) \; = \; \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m}{(d)_m (e)_m}\, \frac{z^m}{m!}\,, \qquad (6.2)$$

and it gets progressively worse in the case of higher generalized hypergeometric series ${}_{p+1}F_p$ with $p \geq 3$. The analytic continuation formulas of the type of Eq. (6.1) are not always known, and those that are known become more and more complicated with increasing p [77–80]. Thus, the efficient and reliable computation of a generalized hypergeometric function ${}_{p+1}F_p$ with $p \geq 2$ via its series expansion (1.3) is still a more or less open problem.

For example, in Theorem 2 of Ref. [80] explicit expressions for the analytic continuation formulas of a ${}_{p+1}F_p$ with $p = 2, 3, \ldots$ around $z = 1$ were constructed. However, these expansions are by some orders of magnitude more complicated than the analogous formula (6.1) for a ${}_2F_1$.

We show here that the CNCT can be very useful if the argument z of a generalized hypergeometric series ${}_{p+1}F_p$ is only slightly smaller than one, even if $z = 1$ is a singularity.

The function ${}_3F_2(1, 3/2, 5; 9/8, 47/8; z)$ has a singularity at $z = 1$ because the sum of the real parts of the numerator parameters is greater than the sum of the real parts of the denominator parameters (see p. 45 of Ref. [74]). Thus, in Table IX we evaluate the function

$$\begin{aligned} & 10^{-4}\,{}_3F_2(1, 3/2, 5; 9/8, 47/8; 0.99999) \\ & \quad = 0.238\;434\;298\;763\;330 \end{aligned} \tag{6.3}$$

in the immediate vicinity of the singularity. With the help of the CNCT, it is nevertheless possible to evaluate this function without any noticeable loss of significant digits.

Insert Table IX here

The function ${}_3F_2(1, 3, 7; 5/2, 14; z)$ does not have a singularity at $z = 1$, i.e., the hypergeometric series (6.2) converges for $z = 1$, albeit very slowly. In Table X we compute the function

$$\begin{aligned} & 10^{-1}\,{}_3F_2(1, 3, 7; 5/2, 14; 0.99999) \\ & \quad = 0.267\;102\;823\;984\;762 \end{aligned} \tag{6.4}$$

with the help of the CNCT.

Insert Table X here

There is another major difference between a Gaussian hypergeometric series ${}_2F_1$ and a generalized hypergeometric series ${}_{p+1}F_p$ with $p \geq 2$. The series (1.2) for a hypergeometric function ${}_2F_1(a, b; c; z)$ with unit argument $z = 1$ converges if $\mathrm{Re}(c) > \mathrm{Re}(a+b)$, and its value is given by the Gauss summation theorem (compare for instance Section 1.7 of Ref. [74]):

$${}_2F_1(a, b; c; 1) \;=\; \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \,. \tag{6.5}$$

The series (1.3) for ${}_{p+1}F_p\bigl(\alpha_1, \ldots, \alpha_{p+1}; \beta_1, \ldots, \beta_p; z\bigr)$ with $p \geq 2$ converges at unit argument $z = 1$ if $\mathrm{Re}(\beta_1 + \ldots + \beta_p) > \mathrm{Re}(\alpha_1 + \ldots + \alpha_{p+1})$ (compare p. 45 of Ref. [74]). In Theorem 1 of Ref. [80], an analogue of the Gauss summation theorem for a generalized hypergeometric series ${}_{p+1}F_p$ with unit argument was formulated. However, this expression is not a simple ratio of gamma functions as in Eq. (6.5), but an infinite series with complicated terms.

Otherwise, there is a variety of different *simple* summation theorems for generalized hypergeometric series ${}_{p+1}F_p$ with unit argument which are ratios of gamma function. However, these summation theorems depend upon the value of p, and are usually only valid for certain values and/or combinations of the parameters α_1, α_2, $\ldots$, α_{p+1} and β_1, β_2, $\ldots$, β_p. A list of these simple summation theorems can be found in Appendix III of Slater's book [74]. The reconstruction of known simple summation theorems for nonterminating generalized hypergeometric series and the construction of new ones with the help of computer algebra systems is discussed in books by Petkovšek, Wilf, and Zeilberger [81] and Koepf [82].

With the help of Watson's summation theorem (p. 245 Ref. [74]) we obtain

$$\begin{aligned} 10^{-1}\,{}_3F_2(1, 3, 7; 5/2, 14; 1) &= \frac{567567\,\pi^2}{20971520} \\ &= 0.267\;108\;047\;538\;428\,. \end{aligned} \tag{6.6}$$

We show in Table XI that this result can also be obtained with the help of the CNCT, starting from the hypergeometric series (6.2).

Insert Table XI here

VII. PRODUCTS OF BESSEL AND HANKEL FUNCTIONS

In this section, we want to apply the CNCT to series whose terms involve more complicated entities and which are more typical of the problems encountered in theoretical physics.

As an example we consider here a series whose terms are products of spherical Bessel and Hankel functions (p. 435 of Ref. [34]):

$$j_l(z) = [\pi/(2z)]^{1/2}\, J_{l+1/2}(z)\,, \tag{7.1}$$

$$h_l^{(1)}(z) = [\pi/(2z)]^{1/2}\, H_{l+1/2}^{(1)}(z)\,. \tag{7.2}$$

Here, the index l is a nonnegative integer, and $J_{l+1/2}$ and $H_{l+1/2}^{(1)}$ are Bessel and Hankel functions, respectively (pp. 358 and 360 of Ref. [34]). In physical applications, the index l usually finds a natural interpretation as an angular momentum quantum number.

Here, we investigate the following model problem:

$$\frac{\exp\bigl(-y[1-r]\bigr)}{y[1-r]} \;=\; -\sum_{l=0}^{\infty} (2l+1)\, j_l(iry)\, h_l^{(1)}(iy)\,, \tag{7.3}$$

where y is positive and $0 < r < 1$. The spherical Bessel and Hankel functions $j_l(iry)$ and $h_l^{(1)}(iy)$ can also be expressed in terms of *modified* spherical Bessel functions. However, we prefer to retain the given notation because there are some inconsistencies in the literature regarding the prefactors to be assigned to the *modified* spherical Bessel functions, whereas the spherical Bessel and Hankel functions are defined consistently in most textbooks.

With the help of known properties of Bessel functions and Legendre polynomials it can be shown easily that the series expansion (7.3) is a special case of the well known addition theorem of the Yukawa potential (p. 107 of Ref. [65]),

$$\frac{\exp(-\gamma w)}{w} \; = \; (r\rho)^{-1/2} \times \sum_{l=0}^{\infty} (2l+1)\, P_l(\cos\phi)\, I_{l+1/2}(\gamma x)\, K_{l+1/2}(\gamma y)\,, \quad (7.4)$$

where $w = (x^2 + y^2 - 2xy\cos\phi)^{1/2}$, $0 < x < y$ and $\gamma > 0$. This addition theorem is also the Green's function of the three-dimensional modified Helmholtz equation.

First, we want to analyze the behaviour of the terms of the series on the right-hand side of Eq. (7.3) if the index l becomes large. The leading orders of the asymptotic expansions of the spherical Bessel and Hankel functions $j_l(iry)$ and $h_l^{(1)}(iy)$ as $l \to \infty$ with r and y fixed can be obtained easily from the series expansion for j_l (Eq. (10.1.2) of Ref. [34] or Eq. (E.17) of Ref. [83]) and from the explicit expression for $h_l^{(1)}$ (Eqs. (10.1.3) and (10.1.16) of Ref. [34] or Eq. (E.18) of Ref. [83]), yielding

$$j_l(iry) \sim \frac{(iry)^l}{(2l+1)!!}\,[1 + \mathrm{O}(1/l)]\,, \quad (7.5)$$

$$h_l^{(1)}(iy) \sim -i\,\frac{(2l-1)!!\mathrm{e}^{-y}}{(iy)^{l+1}}\,[1 + \mathrm{O}(1/l)]\,. \quad (7.6)$$

Thus, we obtain for the leading order of the asymptotic expansion ($l \to \infty$) of the product of the spherical Bessel and the spherical Hankel function:

$$j_l(iry)\,h_l^{(1)}(iy) \;\sim\; -\frac{r^l\mathrm{e}^{-y}}{(2l+1)y}\,[1 + \mathrm{O}(1/l)]\,. \quad (7.7)$$

This asymptotic result shows that the infinite series on the right-hand side of Eq. (7.3) converges only for $|r| < 1$, and that the function $\exp(-y[1-r])/(y[1-r])$ has a singularity at $r = 1$.

Thus, it is interesting to use the CNCT for an evaluation of the series on the right-hand side of Eq. (7.3) in the immediate vicinity of the singularity. This series is similar to the sum over angular momenta encountered in QED bound state calculations (cf. Ref. [30] and Eq. (4.2) in Ref. [83]).

In principle, the spherical Bessel and Hankel functions in the products $j_l(iry)\,h_l^{(1)}(iy)$ can be evaluated recursively. If, however, the CNCT is used, this is not useful. The products belonging to *all* angular momenta l are not needed. Instead, only the products for some *specific* angular momenta have to be evaluated. Accordingly, we prefer to evaluate the spherical Bessel function $j_l(iry)$ from its series expansion (Eq. (10.1.2) of Ref. [34] or Eq. (E.17) of Ref. [83]) and the spherical Hankel function $h_l^{(1)}(iy)$ via its explicit expression (Eq. (10.1.16) of Ref. [34] or Eq. (E.18) of Ref. [83]).

It should be taken into account that the recursive evaluation of the terms of angular momentum decompositions is normally possible only in the case of simple model problems. We shortly outline the applications of the CNCT to a more involved computational problem occurring in QED bound state calculations [30]. The series to be evaluated in the QED calculation is given by Eq. (4.2) in Ref. [83]. This series can be rewritten as

$$S(r,y,t,\gamma) \; = \; J(r,y,t,\gamma) \sum_{\kappa=1}^{\infty} T_\kappa(r,y,t,\gamma)\,, \quad (7.8)$$

where $J(r,y,t,\gamma)$ has a simple mathematical structure. The term $T_\kappa(r,y,t,\gamma)$ ($\kappa = 1, 2, \ldots$) can be rewritten as (see Eq. (5.7) in Ref. [84])

$$T_\kappa(r,y,t,\gamma) \; = \; \sum_{i,j=1}^{2} f_i\left(\frac{r\,y}{a}\right) G_{\mathrm{B},\kappa}^{ij}\left[\frac{r\,y}{a}, \frac{y}{a}, \frac{i}{2}\left(\frac{1}{t} - t\right)\right] \times f_j\left(\frac{y}{a}\right) A_\kappa\left(\frac{r\,y}{a}, \frac{y}{a}\right) - f_{3-i}\left(\frac{r\,y}{a}\right) \times G_{\mathrm{B},\kappa}^{ij}\left[\frac{r\,y}{a}, \frac{y}{a}, \frac{i}{2}\left(\frac{1}{t} - t\right)\right] f_{3-j}\left(\frac{y}{a}\right) A_\kappa^{ij}\left(\frac{r\,y}{a}, \frac{y}{a}\right)\,, \quad (7.9)$$

where

- the functions f_i are the radial Dirac-Coulomb wave functions (see Eq. (A.8) in Ref. [84]),
- the functions $G_{\mathrm{B},\kappa}^{ij}$ are related to the radial components of the relativistic Green function of the bound electron (see Eq. (5.5) in Ref. [84]), and
- the functions A_κ and A_κ^{ij} are related to the angular momentum decomposition of the Green function of the virtual photon and are defined in Eq. (5.8) of Ref. [84].

The quantity a is a scaling variable for the subsequent radial integration over y (see Eq. (4.1) in Ref. [83]). The variable r denotes the ratio of the two radial coordinates ($0 < r < 1$), and t is related to the (complex) energy of the virtual photon ($0 < t < 1$). The dependence on the coupling γ of the electron to the central field, which appears on the left-hand side of Eq. (7.9), is implicitly contained in the functions $G_{\mathrm{B},\kappa}^{ij}$ on the right-hand side of Eq. (7.9) (see Eq. (A.16) in Ref. [84]). The propagator G_{B} in Eq. (7.9) contains the relativistic Dirac-Coulomb Green function, whose radial components are given in Eq. (A.16) in Ref. [84]. Recurrence formulas relating the

$G^{ij}_{\mathrm{B},\kappa}$ for different values of κ are not known. Therefore the terms in the series (7.8) cannot be evaluated by recursion, but each one of these terms can be computed using techniques described in Refs. [30,83]. Note that the angular momentum decomposition of Eq. (7.8) does not correspond to an expansion in terms of the QED perturbation theory. The perturbative parameter in QED is the elementary charge e or the fine structure constant[2] $\alpha = e^2/(4\,\pi)$. The series (7.8) occurs in the evaluation of an energy shift (the Lamb shift) of atomic levels due to the self energy of the electron. The self energy is an effect described by second-order perturbation theory within the framework of QED. The asymptotic behaviour of the terms T_κ in Eq. (7.8),

$$T_\kappa \sim \frac{r^{2\,\kappa}}{\kappa}\left[1 + \mathrm{O}\left(\frac{1}{\kappa}\right)\right], \qquad \kappa \to \infty\,, \tag{7.10}$$

is similar to that of the model series (7.7) and to that of the Li_1-series discussed in Sec. V (see Eq. (5.5) and the results presented in Table V). Slow convergence of the series in Eq. (7.8) is observed in the limit $r \to 1$.

Returning to our model problem given in Eq. (7.3), we consider here the example of $r = 0.9999$ and $y = 0.7$. This yields

$$\begin{aligned} & 10^{-5} \sum_{l=0}^{\infty} (2\,l+1)\, j_l(i\,0.9999 \times 0.7)\, h_l^{(1)}(i\,0.7) \\ & \quad = -0.142\,847\,143\,207\,135\,. \end{aligned} \tag{7.11}$$

Evaluation of the series over the spherical Bessel and Hankel functions via the CNCT is presented in Table XII. If this series is evaluated by adding up its terms, then about 450 000 products of Bessel and Hankel functions have to be evaluated for a relative accuracy of 10^{-16} in the final result. This is to be compared with the approximately 300 evaluations of products of Bessel and Hankel functions which are needed to compute the 25 Van Wijngaarden terms presented in Table XII with the necessary accuracy. According to our experience, the use of the CNCT in QED bound state calculations reduces the amount of computer time for slowly convergent series of the type of Eq. (7.8) by three orders of magnitude.

Insert Table XII here

VIII. SUMMARY AND CONCLUSIONS

Our approach for the acceleration of the convergence of monotone series consists of two steps. In the first step, the Van Wijngaarden condensation transformation, Eqs. (2.2) - (2.4), is applied to the original monotone series. In the second step, the convergence of the resulting alternating series (2.3) is accelerated with the help of suitable nonlinear sequence transformations.

Daniel [21] showed that the transformed alternating series (2.3) converges under relatively mild conditions to the same limit as the original series. The condensation transformation cannot solve the problems due to slow convergence on its own since the transformed alternating series does not converge more rapidly than the original series (see Eq. (3.2)). However, the convergence of alternating series can be accelerated much more effectively than the convergence of the original monotone series. Moreover, the transformation of alternating series by sequence transformations is numerically a relatively stable process, whereas the direct transformation of monotone series inevitably leads to the loss of accuracy due to round-off.

Many sequence transformations are known which are in principle capable of accelerating the convergence of alternating series [1,2,9]. However, their efficiency varies considerably. In all numerical examples studied in the context of this paper, we found that certain sequence transformations [9,12] (which also use explicit estimates for the truncation errors as input data) are significantly more powerful than other, better known sequence transformations as for instance Wynn's epsilon algorithm [11] or the Euler transformation [35] (which only use the partial sums of the infinite series to be transformed as input data). Consequently, we use in this article only the sequence transformations $\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n)$ and $\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n)$ defined in Eqs. (3.8) and (3.9) in combination with the truncation error estimate (3.12). Those properties of these sequence transformations, which are needed to apply them successfully in a convergence acceleration process, are described in Section III. In Appendix A, we explain why these transformations are much more effective in the case of alternating series than in the case of monotone series, and in Appendix B we discuss exactness properties of these transformations.

We consistently observed that the combined transformation is a remarkably stable numerical process and entails virtually no loss of numerical significance at intermediate stages. The evaluation of the condensed series (2.4) is a computationally simple task, provided that the evaluation of terms of sufficiently high indices in the original series is feasible. The nonlinear sequence transformations used in the second step are also remarkably stable. The final results, which we present in the tables, indicate that a relative accuracy of 10^{-14} can be achieved with floating point arithmetic of 16 decimal digits. Moreover, our method remains stable in higher transformation orders.

In Sections IV - VI we demonstrate that our approach is very useful for the computation of special functions which are defined by logarithmically convergent series. Our approach also works very well if the argument of a power series with monotone coefficients is very close to

[2] We use natural units as it is customary for QED bound state calculations ($\hbar = c = \epsilon_0 = 1$).

the boundary of the circle of convergence (even in the vicinity of singularities).

In Section IV, we treat the Dirichlet series (1.4) for the Riemann zeta function which because of its simplicity is excellently suited to illustrate the mechanism of our combined transformation. The application of the Van Wijngaarden transformation to the logarithmically convergent series (1.4) yields the known alternating series (4.3) which does not converge more rapidly than the original monotone series (1.4). However, the alternating series can be transformed effectively by the sequence transformations (3.13) or (3.14) in the case of slow convergence or even summed in the case of divergence. Consequently, the Riemann zeta function can be evaluated effectively and reliably even if its argument z is only slightly larger than one (Table II), if both the monotone series (1.4) and the alternating series (4.3) diverge (Table III), or if the argument z is complex (Table IV).

In Section V we treat the Lerch transcendent $\Phi(z, s, \alpha)$ which contains many other special functions as special cases, for example the Riemann zeta function or the polylogarithms. In Tables V - VIII we show that the combined transformation makes it possible to evaluate $\Phi(z, s, \alpha)$ effectively and reliable via the power series (5.1) even if z is very close to the boundary of the circle of convergence.

In Section VI we discuss the evaluation of a nonterminating generalized hypergeometric series ${}_{p+1}F_p$ via its defining power series (1.3) which converges for $|z| < 1$. In the case of a Gaussian hypergeometric series ${}_2F_1$, explicit analytic continuation formulas are known which transform a hypergeometric series with argument z into the sum of two hypergeometric series ${}_2F_1$ with argument $1-z$. Thus, if the convergence of a hypergeometric series ${}_2F_1$ is slow because its argument z is only slightly smaller then one, then the two transformed hypergeometric series ${}_2F_1$ with argument $1-z$ will converge rapidly. Moreover, if a Gaussian hypergeometric series ${}_2F_1$ with unit argument $z = 1$ converges, then its value is given by the Gauss summation theorem (6.5). In the case of a generalized hypergeometric series ${}_{p+1}F_p$ with $p \geq 2$, the situation is much more difficult. Explicit analytic continuation formulas are either unknown or they become increasingly complicated with increasing p. Similarly, simple analogues for the Gauss summation theorem (6.5) exist only for certain values of p and for certain combinations of the parameters of the ${}_{p+1}F_p$. In Tables IX - XI we show that a generalized hypergeometric series ${}_{p+1}F_p$ ($p \geq 2$) with an argument z that is only slightly smaller than one or with unit argument ($z = 1$) can be computed effectively and reliably with the help of the combined transformation.

Partial wave decompositions of Green's functions, which occur for example in quantum electrodynamic bound state calculations, entail more complex mathematical entities than series expansions for special functions. Nevertheless, we show in Section VII that the combined transformation can be applied successfully to these partial wave decompositions. The series over products of spherical Bessel and Hankel functions considered in this paper serves as a model problem for the angular momentum decomposition of more complex Green's functions, as for example the relativistic Green's function of the bound electron [84]. In the model problem studied here as well as in a recent evaluation of self energy corrections in bound systems [30], we observed a reduction in computer time by three orders of magnitude. Thus, the combined transformations makes extensive and highly accurate calculations feasible in situations that could otherwise be too time-consuming.

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APPENDIX A: ON THE EFFICIENCY OF THE TRANSFORMATION OF ALTERNATING AND MONOTONE SERIES

In the case of the Levin transformation $\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n)$, Eq. (3.8), and the closely related Weniger transformation $\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n)$, Eq. (3.9), it is relatively easy to understand why alternating series can be transformed much more effectively than monotone series. These two transformations are both special cases of the following transformation which is characterized by the positive weights $w_k(n)$:

$$\mathcal{T}_k^{(n)}\big(w_k(n); s_n, \omega_n\big) \; = \; \frac{\Delta^k \{w_k(n) s_n/\omega_n\}}{\Delta^k \{w_k(n)/\omega_n\}} \, . \qquad \text{(A1)}$$

If $w_k(n) = (n+\beta)^{k-1}$, we obtain Levin's transformation, and it $w_k(n) = (n+\beta)_{k-1}$, we obtain Weniger's transformation.

Since the difference operator Δ^k is linear, Eq. (A1) can also be rewritten as follows:

$$\mathcal{T}_k^{(n)}\big(w_k(n); s_n, \omega_n\big) = s \, + \, \frac{\Delta^k \{w_k(n)[s_n - s]/\omega_n\}}{\Delta^k \{w_k(n)/\omega_n\}} \, . \quad \text{(A2)}$$

Obviously, the sequence transformation $\mathcal{T}_k^{(n)}$ converges to the (generalized) limit s of the input sequence $\{s_n\}_{n=0}^{\infty}$ if the remainder estimates $\{\omega_n\}_{n=0}^{\infty}$ can be chosen in such a way that the ratio on the right-hand side becomes negligibly small.

Loosely speaking, this means that we have to choose the remainder estimates in such a way that the numerator of the ratio on the right-hand side of Eq. (A2) becomes as small as possible and the denominator becomes as large as possible.

If we can find remainder estimates such that

$$s_n - s \;=\; \omega_n \bigl[c + O(n^{-1})\bigr] \,, \qquad n \to \infty \,, \tag{A3}$$

then the weighted difference operator $\Delta^k w_k(n)$ will make the numerator small, no matter whether the input data $\{s_n\}_{n=0}^{\infty}$ are the partial sums of an alternating or of a monotone series.

The situation is quite different in the case of the denominator. Application of the weighted difference operator $\Delta^k w_k(n)$ to $1/\omega_n$ yields

$$\begin{aligned} &\Delta^k \{ w_k(n)/\omega_n \} \\ &\qquad = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{w_k(n+j)}{\omega_{n+j}} \,. \end{aligned} \tag{A4}$$

If the remainder estimates all have the same sign, the denominator will also become relatively small because of cancellation due to differencing (hopefully not as small as the numerator because then the transformation process would not converge). This cancellation process is also the major source of numerical instabilities which are magnified since they occur in the denominator.

If the remainder estimates are strictly alternating in sign, $\omega_n = (-1)^n |\omega_n|$, then we obtain ($w_k(n)$ is by assumption positive)

$$\Delta^k \{ w_k(n)/\omega_n \} \;=\; (-1)^{k+n} \sum_{j}^{k} \binom{k}{j} \frac{w_k(n+j)}{|\omega_{n+j}|} \,. \tag{A5}$$

Thus, all terms of the denominator sum have the same sign, and there is no cancellation as in the case of monotone remainder estimates and also no source of numerical instabilities. Consequently, the denominator becomes relatively large which improves convergence.

As an example, we apply the transformation (A1) to the monotone model sequence

$$s_n \;=\; s + \frac{c_0}{(n+1)^{\alpha}} + \frac{c_1}{(n+1)^{\alpha+1}} + \ldots \,, \tag{A6}$$

as well as to the alternating model sequence

$$\begin{aligned} t_n \;=\; & t \\ & + (-1)^n \left\{ \frac{c_0}{(n+1)^{\alpha}} + \frac{c_1}{(n+1)^{\alpha+1}} + \ldots \right\} \end{aligned} \tag{A7}$$

and derive asymptotic ($n \to \infty$) transformation error estimates.

For that purpose, we assume $\alpha > 0$ – which implies that the two model sequences converge to their limits s and t, respectively – and $w_k(n) = O\bigl(n^{k-1}\bigr)$ as $n \to \infty$. In the case of the monotone input sequence $\{s_n\}_{n=0}^{\infty}$, we choose $\omega_n = (n+1)^{-\alpha}$, and in the case of the alternating input sequence $\{t_n\}_{n=0}^{\infty}$, we choose $\omega_n = (-1)^n (n+1)^{-\alpha}$. Then, we obtain for the numerators

$$\begin{aligned} &\Delta^k \left\{ \frac{w_k(n) [s_n - s]}{(n+1)^{-\alpha}} \right\} \\ &\qquad = \Delta^k \left\{ \frac{w_k(n) [t_n - t]}{(-1)^n (n+1)^{-\alpha}} \right\} \;=\; O\bigl(n^{-k-1}\bigr) \,. \end{aligned} \tag{A8}$$

The estimates for the denominators differ considerably. In the case of the monotone input sequence (A6), we obtain

$$\Delta^k \left\{ \frac{w_k(n)}{(n+1)^{-\alpha}} \right\} \;=\; O\bigl(n^{\alpha-1}\bigr) \,, \tag{A9}$$

which in combination with Eq. (A8) yields the following asymptotic transformation error estimate as $n \to \infty$:

$$\frac{\mathcal{T}_k^{(n)}\bigl(w_k(n); s_n, (n+1)^{-\alpha}\bigr) - s}{s_n - s} \;=\; O\bigl(n^{-k}\bigr) \,. \tag{A10}$$

In the case of the alternating input sequence (A7), we exploit the fact that the first term with $j = 0$ in the sum on the right-hand side of Eq. (A5) is smaller in magnitude than the whole sum, yielding

$$\begin{aligned} &\frac{1}{|\Delta^k \{ w_k(n)/[(-1)^n (n+1)^{-\alpha}] \}|} \\ &\qquad \le \frac{1}{|w_k(n)/[(-1)^n (n+1)^{-\alpha}]|} \,. \end{aligned} \tag{A11}$$

Of course, we could also try to construct more sophisticated estimates for the denominators. However, the relatively crude estimate (A11) suffices for our purposes since it implies in combination with Eq. (A8)

$$\frac{\mathcal{T}_k^{(n)}\bigl(w_k(n); t_n, (-1)^n (n+1)^{-\alpha}\bigr) - t}{t_n - t} \;=\; O\bigl(n^{-2k}\bigr) \,. \tag{A12}$$

The two transformation error estimates (A10) and (A12), which both hold as $n \to \infty$, show that there is a substantial difference between the transformation of monotone and alternating series. There is a considerable amount of evidence that this conclusion is actually generally true, i.e., also in the case of Padé approximants and other sequence transformations.

APPENDIX B: EXACTNESS RESULTS

In this Appendix, we analyze exactness properties of the Levin transformation $\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n)$, Eq. (3.8),

and of the Weniger transformation $\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n)$, Eq. (3.9), and their variants $d_k^{(n)}(\beta, \mathbf{S}_n)$, Eq. (3.13), and $\delta_k^{(n)}(\beta, \mathbf{S}_n)$, Eq. (3.14), which both use the remainder estimate (3.12) proposed by Smith and Ford [37].

For that purpose we introduce in Eq. (A2) the remainder $r_n = s_n - s$ and obtain

$$\mathcal{T}_k^{(n)}\bigl(w_k(n); s_n, \omega_n\bigr) = s + \frac{\Delta^k\{w_k(n) r_n/\omega_n\}}{\Delta^k\{w_k(n)/\omega_n\}} . \qquad \text{(B1)}$$

Obviously, the general sequence transformation $\mathcal{T}_k^{(n)}$, Eq. (A1), which contains $\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n)$ and $\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n)$ as special cases, is exact for a given input sequence $\{s_n\}_{n=0}^{\infty}$ if the difference operator Δ^k annihilates $w_k(n) r_n/\omega_n$ but not $w_k(n)/\omega_n$.

Thus, if we want to prove the exactness of $\mathcal{T}_k^{(n)}$ for some sequence $\{s_n\}_{n=0}^{\infty}$, we need to know explicit expressions for the remainders $\{r_n\}_{n=0}^{\infty}$ of the input sequence as functions of n.

In the case of the alternating series (4.3) for the Riemann zeta function with zero or negative integral argument, this can be accomplished easily. Replacing z by $-l$ in Eq. (4.3) with $l = 0, 1, 2, \ldots$ we obtain

$$\zeta(-l) = \frac{1}{1 - 2^{l+1}} \sum_{j=0}^{\infty} (-1)^j (j+1)^l . \qquad \text{(B2)}$$

For $l = 0$, this series is the negative of the geometric series $1/(1+x) = \sum_{j=0}^{\infty} (-x)^j$ at $x = 1$. According to Theorem 12-9 of Ref. [9], the sequence transformations $\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n)$ and $\mathcal{S}_k^{(n)}(\beta, s_n, \omega_n)$ are exact for the geometric series if the first term neglected in the partial sum is used as remainder estimate (which corresponds to the remainder estimate (3.12)). Thus, the exactness of these sequence transformations for the infinite series (B2) has to be analyzed only for $l \ge 1$.

For the determination of its truncation error with arbitrary integral $l \ge 1$, we rewrite the infinite series (B2) as follows:

$$\begin{aligned} \sum_{j=0}^{\infty} (-1)^j (j+1)^l &= \sum_{j=0}^{n} (-1)^j (j+1)^l + \sum_{j=n+1}^{\infty} (-1)^j (j+1)^l && \text{(B3)} \\ &= \sum_{j=0}^{n} (-1)^j (j+1)^l \\ &\quad + (-1)^{n+1} \sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)^l . && \text{(B4)} \end{aligned}$$

The infinite series on the right-hand side of Eq. (B4) obviously diverges. However, it can be summed easily since it can be represented as a derivative of the geometric series $1/(1+x) = \sum_{j=0}^{\infty} (-x)^j$ at $x = 1$:

$$\begin{aligned} \sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)^l &= \lim_{x \to 1-} \left\{ \sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)^l x^{n+\nu+1} \right\} && \text{(B5)} \\ &= \lim_{x \to 1-} \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}x} x \right)^l \sum_{\nu=0}^{\infty} (-1)^{\nu} x^{n+\nu+1} \right\} && \text{(B6)} \\ &= \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}x} x \right)^l \frac{x^{n+1}}{1+x} \right\} \Bigg|_{x=1} . && \text{(B7)} \end{aligned}$$

By inserting this into Eq. (B2) we obtain the following representation of $\zeta(-l)$ as a partial sum plus an explicit expression for the truncation error:

$$\begin{aligned} \zeta(-l) = \frac{1}{1 - 2^{l+1}} \Biggl\{ & \sum_{j=0}^{n} (-1)^j (j+1)^l \\ & + (-1)^{n+1} \left[\left(\frac{\mathrm{d}}{\mathrm{d}x} x \right)^l \frac{x^{n+1}}{1+x} \right] \Bigg|_{x=1} \Biggr\} . \qquad \text{(B8)} \end{aligned}$$

If we set in Eq. (B8) $n = -1$, then the partial sum is an empty sum and vanishes, yielding the following explicit expression for the Riemann zeta function with zero or negative integer argument:

$$\zeta(-l) = \frac{1}{1 - 2^{l+1}} \left[\left(\frac{\mathrm{d}}{\mathrm{d}x} x \right)^l \frac{1}{1+x} \right] \Bigg|_{x=1} . \qquad \text{(B9)}$$

Equivalent explicit expressions can be obtained via the substitution $y = \ln(x)$. Obviously, we have

$$\left[\frac{\mathrm{d}}{\mathrm{d}y} + 1 \right] f\bigl(\mathrm{e}^y\bigr) \Bigg|_{y=\ln(x)} = \frac{\mathrm{d}}{\mathrm{d}x} \bigl[x\, f(x) \bigr] . \qquad \text{(B10)}$$

In this way, we obtain:

$$\begin{aligned} \zeta(-l) &= \frac{1}{1 - 2^{l+1}} \left\{ \left[\frac{\mathrm{d}}{\mathrm{d}y} + 1 \right]^l \frac{1}{1+\mathrm{e}^y} \right\} \Bigg|_{y=0} && \text{(B11)} \\ &= \frac{1}{1 - 2^{l+1}} \Biggl\{ \sum_{j=0}^{n} (-1)^j (j+1)^l \\ &\quad + (-1)^{n+1} \left[\left(\frac{\mathrm{d}}{\mathrm{d}y} + 1 \right)^l \frac{\mathrm{e}^{(n+1)y}}{1+\mathrm{e}^y} \right] \Bigg|_{y=0} \Biggr\} . && \text{(B12)} \end{aligned}$$

Combination of Eqs. (4.11b), (B9), and (B11) yields the following expressions for the Bernoulli numbers with even indices:

$$\begin{aligned} B_{2l} &= \frac{-2l}{1 - 2^{2l}} \left[\left(\frac{\mathrm{d}}{\mathrm{d}x} x \right)^{2l-1} \frac{1}{1+x} \right] \Bigg|_{x=1} && \text{(B13)} \\ &= \frac{-2l}{1 - 2^{2l}} \left\{ \left[\frac{\mathrm{d}}{\mathrm{d}y} + 1 \right]^{2l-1} \frac{1}{1+\mathrm{e}^y} \right\} \Bigg|_{y=0} . && \text{(B14)} \end{aligned}$$

It follows from Eq. (B1) that the general sequence transformation $\mathcal{T}_k^{(n)}$, Eq. (A1), is exact for some input sequence $\{s_n\}_{n=0}^{\infty}$ if the expression $w_k(n) r_n / \omega_n$ with $r_n = s_n - s$ is a polynomial of degree $k-1$ in n. Thus, Eq. (B7) implies that in the case of the Weniger transformation $\delta_n^{(0)}\bigl(1, \mathbf{S}_0\bigr)$, Eq. (3.14), which uses the remainder estimate (3.12), we have to show that the ratio

$$\frac{(n+1)_{k-1} \sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)^l}{(-1)^{n+1}\,(n+2)^l} \tag{B15}$$

is for sufficiently large values of k a polynomial of degree $k-1$ in n. For $l = 1$, 2, 3, and 4, Eq. (B7) yields the following explicit expressions:

$$\sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)^1 \;=\; \frac{2n+3}{4}\,, \tag{B16}$$

$$\sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)^2 \;=\; \frac{(n+1)(n+2)}{2}\,, \tag{B17}$$

$$\sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)^3 = \frac{(2n+3)(2n^2+6n+3)}{8}\,, \tag{B18}$$

$$\sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)^4 = \frac{(n+1)(n+2)(n^2+3n+1)}{2}\,. \tag{B19}$$

Since $(n+1)_{k-1} = (n+1)(n+2)\ldots(n+k-1)$, the ratio (B15) is for $l = 1$, 2 and $k \ge 3$ a polynomial of degree $k-1$ in n, whereas for $l = 3$, 4 it is a rational expression in n which is not annihilated by the operator Δ^k.

Thus, the exactness of the transformation $\delta_n^{(0)}\bigl(1, \mathbf{S}_0\bigr)$ for $\zeta(-1)$ as in Table III, or for $\zeta(-2)$ is more or less accidental. If, however, we choose in the Levin transformation $d_n^{(0)}\bigl(\beta, \mathbf{S}_0\bigr)$, Eq. (3.13), with $\beta = 2$ instead of the usual $\beta = 1$, we find that the corresponding ratio

$$\frac{(n+2)^{k-1} \sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)^l}{(-1)^{n+1}\,(n+2)^l} \tag{B20}$$

is a polynomial of degree $k-1$ in n for all $l = 0$, 1, 2, ... and $k \ge l+1$. This follows at once from the fact that the differential operator in Eq. (B7) produces a polynomial of degree l in n.

In the same way, it can be shown that the Weniger transformation $\delta_n^{(0)}\bigl(\beta, \mathbf{S}_0\bigr)$, Eq. (3.14), with $\beta = 2$ is exact for the hypergeometric series for all $l = 0$, 1, 2, ... and for $k \ge l+1$:

$$l!\, {}_1F_0(l+1; -1) \;=\; \sum_{j=0}^{\infty} (-1)^j (j+1)_l \;=\; \frac{l!}{2^{l+1}}\,. \tag{B21}$$

This can also be rewritten as follows:

$$\begin{aligned} l!\, {}_1F_0(l+1; -1) &= \sum_{j=0}^{n} (-1)^j (j+1)_l \\ &\quad + (-1)^{n+1} \sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)_l & \text{(B22)} \\ &= \sum_{j=0}^{n} (-1)^j (j+1)_l \\ &\quad + (-1)^{n+1}\, (n+2)_l \, {}_2F_1(1, n+l+2; n+2; -1) & \text{(B23)} \\ &= \sum_{j=0}^{n} (-1)^j (j+1)_l \\ &\quad + (-1)^{n+1} \sum_{j=0}^{l} (-l)_j\, (n+j+2)_{l-j}\, (1/2)^{l+1}\,. & \text{(B24)} \end{aligned}$$

In this case, the corresponding ratio

$$\frac{(n+2)_{k-1} \sum_{\nu=0}^{\infty} (-1)^{\nu} (n+\nu+2)_l}{(-1)^{n+1}\,(n+2)_l} \tag{B25}$$

is a polynomial of degree $k-1$ in n for all $l = 0$, 1, 2, ... and for $k \ge l+1$. This follows at once from the fact that the second sum on the right-hand side of Eq. (B24) is a polynomial of degree l in n.

However, the value $\beta = 2$ in the Eqs. (3.13) and (3.14) cannot significantly improve convergence for the zeta function. We also investigated the performance of the transformations $d_n^{(0)}\bigl(2, \mathbf{S}_n\bigr)$ and $\delta_n^{(0)}\bigl(2, \mathbf{S}_n\bigr)$. Except for the cases of $z = -1$ or $z = -2$ in which the Levin transformation becomes exact with $\beta = 2$, the transformations $d_n^{(0)}\bigl(2, \mathbf{S}_n\bigr)$ and $\delta_n^{(0)}\bigl(2, \mathbf{S}_n\bigr)$ yield results which differ only marginally from the results obtained by $d_n^{(0)}\bigl(1, \mathbf{S}_n\bigr)$ and $\delta_n^{(0)}\bigl(1, \mathbf{S}_n\bigr)$, which are presented in Tables II - IV.

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TABLES

TABLE I. Demonstration that Van Wijngaarden's transformation, Eqs. (2.2) - (2.4), is a rearrangement of the original series $\sum_{k=0}^{\infty} a(k)$.

$(-1)^j \mathbf{A}_j$	Leading terms of the original series						
$\mathbf{A}_0$	$a(0)$	$+2\,a(1)$		$+4\,a(3)$		$+$	$\dots$
$-\mathbf{A}_1$		$-a(1)$		$-2\,a(3)$		$-$	$\dots$
$\mathbf{A}_2$			$+a(2)$			$+$	$\dots$
$-\mathbf{A}_3$				$-a(3)$		$-$	$\dots$
$\mathbf{A}_4$					$+a(4)$	$-$	$\dots$
$\sum_{j=0}^{\infty} (-1)^j \mathbf{A}_j$	$a(0)$	$+a(1)$	$+a(2)$	$+a(3)$	$+a(4)$	$+$	$\dots$

TABLE II. Evaluation of the Riemann zeta function of argument $z = 1.01$ by accelerating the convergence of the alternating series (4.3). The result is given as $10^{-3}\,\zeta(1.01)$.

n	$\mathbf{S}_n$	$\mathbf{E}_n$	$d_n^{(0)}\big(1, \mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1, \mathbf{S}_0\big)$
0	0.144 770 081 711 084	0.144 770 081 711 084	0.144 770 081 711 084	0.144 770 081 711 084
1	0.072 885 040 855 542	0.090 606 301 069 428	0.101 569 133 143 252	0.101 569 133 143 252
2	0.120 614 482 322 980	0.096 697 481 252 857	0.100 456 642 533 059	0.100 456 642 533 059
3	0.084 920 235 019 068	0.098 985 546 018 036	0.100 587 783 459 042	0.100 579 332 613 649
4	0.113 411 984 373 518	0.099 901 837 494 047	0.100 577 428 866 203	0.100 578 083 572 921
5	0.089 712 109 307 161	0.100 283 957 662 399	0.100 577 954 415 585	0.100 577 949 566 834
6	0.109 994 997 614 328	0.100 447 835 715 031	0.100 577 944 116 204	0.100 577 943 567 122
7	0.092 271 153 050 434	0.100 519 572 572 454	0.100 577 943 249 050	0.100 577 943 346 150
8	0.108 007 136 313 467	0.100 551 470 653 282	0.100 577 943 342 049	0.100 577 943 338 734
9	0.093 859 665 080 579	0.100 565 830 599 811	0.100 577 943 338 553	0.100 577 943 338 503
10	0.106 708 750 240 925	0.100 572 360 153 981	0.100 577 943 338 482	0.100 577 943 338 497
11	0.094 940 666 205 327	0.100 575 353 801 276	0.100 577 943 338 498	0.100 577 943 338 497
12	0.105 794 821 569 601	0.100 576 735 870 616	0.100 577 943 338 497	0.100 577 943 338 497
13	0.095 723 429 487 784	0.100 577 377 707 430	0.100 577 943 338 497	0.100 577 943 338 497
14	0.105 116 912 361 076	0.100 577 677 299 954	0.100 577 943 338 497	0.100 577 943 338 497
15	0.096 316 203 847 728	0.100 577 817 763 434	0.100 577 943 338 497	0.100 577 943 338 497
exact	0.100 577 943 338 497	0.100 577 943 338 497	0.100 577 943 338 497	0.100 577 943 338 497

TABLE III. Evaluation of the Riemann zeta function of argument $z = -1$ by summing the divergent alternating series (4.3). The result is given as $10\,\zeta(-1)$.

n	$\mathbf{S}_n$	$d_n^{(0)}\big(1, \mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1, \mathbf{S}_0\big)$
0	−3.333 333 333 333 333	−3.333 333 333 333 333	−3.333 333 333 333 333
1	3.333 333 333 333 333	−0.666 666 666 666 667	−0.666 666 666 666 667
2	−6.666 666 666 666 667	−0.860 215 053 763 441	−0.860 215 053 763 441
3	6.666 666 666 666 667	−0.830 449 826 989 619	−0.833 333 333 333 333
4	−10.000 000 000 000 000	−0.833 557 890 954 819	−0.833 333 333 333 333
5	10.000 000 000 000 000	−0.833 319 627 418 409	−0.833 333 333 333 333
6	−13.333 333 333 333 333	−0.833 334 020 666 741	−0.833 333 333 333 333
7	13.333 333 333 333 333	−0.833 333 304 093 535	−0.833 333 333 333 333
8	−16.666 666 666 666 667	−0.833 333 334 413 139	−0.833 333 333 333 333
9	16.666 666 666 666 667	−0.833 333 333 298 109	−0.833 333 333 333 333
10	−20.000 000 000 000 000	−0.833 333 333 334 362	−0.833 333 333 333 333
11	20.000 000 000 000 000	−0.833 333 333 333 306	−0.833 333 333 333 333
12	−23.333 333 333 333 333	−0.833 333 333 333 334	−0.833 333 333 333 333
13	23.333 333 333 333 330	−0.833 333 333 333 333	−0.833 333 333 333 333
14	−26.666 666 666 666 670	−0.833 333 333 333 333	−0.833 333 333 333 333
15	26.666 666 666 666 670	−0.833 333 333 333 333	−0.833 333 333 333 333
exact	−0.833 333 333 333 333	−0.833 333 333 333 333	−0.833 333 333 333 333

TABLE IV. Evaluation of $\zeta(1/2 + 13.7\ i)$ with the CNCT.

n	$d_n^{(0)}\big(1, \mathbf{S}_0\big)$			$\delta_n^{(0)}\big(1, \mathbf{S}_0\big)$		
0	0.414 107 543 949 134	+	0.017 316 297 125 790 i	0.414 107 543 949 134	+	0.017 316 297 125 790 i
1	0.575 871 239 097 112	−	0.042 690 435 565 758 i	0.575 871 239 097 112	−	0.042 690 435 565 758 i
2	0.523 424 912 174 020	+	0.152 835 043 959 961 i	0.567 958 887 269 553	+	0.129 913 386 486 220 i
3	0.481 953 715 196 159	−	0.288 400 086 031 923 i	0.474 129 917 411 618	−	0.168 775 423 138 291 i
4	0.012 442 899 246 184	−	0.237 603 260 694 125 i	−0.180 827 868 994 142	−	0.367 542 940 737 051 i
5	0.123 074 021 609 358	−	0.316 357 718 264 423 i	0.126 392 529 409 594	−	0.290 235 127 404 228 i
6	0.105 377 569 236 175	−	0.313 246 538 788 829 i	0.107 386 234 787 298	−	0.317 087 400 005 856 i
7	0.107 635 288 132 001	−	0.312 843 180 304 712 i	0.107 124 241 668 490	−	0.312 622 662 817 549 i
8	0.107 429 326 957 578	−	0.312 999 708 812 577 i	0.107 489 121 873 498	−	0.312 978 336 617 038 i
9	0.107 438 933 679 469	−	0.312 974 184 308 675 i	0.107 436 183 812 222	−	0.312 980 180 873 835 i
10	0.107 439 640 888 613	−	0.312 976 813 188 762 i	0.107 439 393 557 558	−	0.312 976 229 866 877 i
11	0.107 439 434 156 190	−	0.312 976 661 398 796 i	0.107 439 488 450 447	−	0.312 976 678 675 422 i
12	0.107 439 457 152 512	−	0.312 976 659 181 027 i	0.107 439 453 065 793	−	0.312 976 661 841 904 i
13	0.107 439 455 840 151	−	0.312 976 660 717 255 i	0.107 439 455 877 677	−	0.312 976 660 319 609 i
14	0.107 439 455 825 365	−	0.312 976 660 547 552 i	0.107 439 455 848 368	−	0.312 976 660 568 977 i
15	0.107 439 455 836 355	−	0.312 976 660 556 014 i	0.107 439 455 833 989	−	0.312 976 660 556 440 i
16	0.107 439 455 835 269	−	0.312 976 660 556 233 i	0.107 439 455 835 348	−	0.312 976 660 556 072 i
17	0.107 439 455 835 311	−	0.312 976 660 556 157 i	0.107 439 455 835 317	−	0.312 976 660 556 169 i
18	0.107 439 455 835 313	−	0.312 976 660 556 163 i	0.107 439 455 835 312	−	0.312 976 660 556 163 i
19	0.107 439 455 835 313	−	0.312 976 660 556 163 i	0.107 439 455 835 313	−	0.312 976 660 556 163 i
20	0.107 439 455 835 313	−	0.312 976 660 556 163 i	0.107 439 455 835 313	−	0.312 976 660 556 163 i
21	0.107 439 455 835 313	−	0.312 976 660 556 163 i	0.107 439 455 835 313	−	0.312 976 660 556 163 i
22	0.107 439 455 835 313	−	0.312 976 660 556 163 i	0.107 439 455 835 313	−	0.312 976 660 556 163 i
23	0.107 439 455 835 313	−	0.312 976 660 556 163 i	0.107 439 455 835 313	−	0.312 976 660 556 163 i
24	0.107 439 455 835 313	−	0.312 976 660 556 163 i	0.107 439 455 835 313	−	0.312 976 660 556 163 i
25	0.107 439 455 835 313	−	0.312 976 660 556 163 i	0.107 439 455 835 313	−	0.312 976 660 556 163 i
exact	0.107 439 455 835 313	−	0.312 976 660 556 163 i	0.107 439 455 835 313	−	0.312 976 660 556 163 i

TABLE V. Evaluation of $10^{-2}\,\mathrm{Li}_1(0.99999) = -10^{-2}\,\ln(0.00001)$ with the CNCT.

n	$\mathbf{S}_n$	$d_n^{(0)}\big(1, \mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1, \mathbf{S}_0\big)$
0	0.162 768 973 713 089	0.162 768 973 713 089	0.162 768 973 713 089
1	0.086 384 436 856 544	0.116 225 388 785 336	0.116 225 388 785 336
2	0.135 357 615 351 010	0.114 995 657 006 664	0.114 995 657 006 664
3	0.099 665 296 922 988	0.115 140 148 148 939	0.115 131 002 772 470
4	0.127 575 317 431 702	0.115 128 665 188 679	0.115 129 400 970 919
5	0.104 755 344 851 635	0.115 129 271 216 942	0.115 129 261 517 373
6	0.123 997 631 730 577	0.115 129 254 851 082	0.115 129 254 830 226
7	0.107 401 422 517 315	0.115 129 254 746 908	0.115 129 254 725 403
8	0.121 964 815 378 398	0.115 129 254 612 355	0.115 129 254 664 668
9	0.109 009 755 125 042	0.115 129 254 647 759	0.115 129 254 647 723
10	0.120 662 087 466 136	0.115 129 254 650 661	0.115 129 254 648 068
11	0.110 085 384 510 686	0.115 129 254 650 507	0.115 129 254 649 602
12	0.119 759 673 589 890	0.115 129 254 649 358	0.115 129 254 649 772
13	0.110 852 765 866 205	0.115 129 254 649 714	0.115 129 254 649 711
14	0.119 099 529 780 039	0.115 129 254 649 727	0.115 129 254 649 700
15	0.111 426 375 175 159	0.115 129 254 649 696	0.115 129 254 649 702
16	0.118 596 725 346 641	0.115 129 254 649 702	0.115 129 254 649 702
17	0.111 870 534 473 655	0.115 129 254 649 703	0.115 129 254 649 702
18	0.118 201 666 639 655	0.115 129 254 649 702	0.115 129 254 649 702
19	0.112 224 086 515 227	0.115 129 254 649 702	0.115 129 254 649 702
20	0.117 883 505 752 715	0.115 129 254 649 702	0.115 129 254 649 702
exact	0.115 129 254 649 702	0.115 129 254 649 702	0.115 129 254 649 702

TABLE VI. Evaluation of $10^{-1}\,\mathrm{Li}_2(0.99999)$ with the CNCT.

n	$\mathbf{S}_n$	$d_n^{(0)}\big(1, \mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1, \mathbf{S}_0\big)$
0	0.199 982 280 324 442	0.199 982 280 324 442	0.199 982 280 324 442
1	0.149 990 640 162 221	0.165 371 886 328 955	0.165 371 886 328 955
2	0.172 207 484 145 972	0.164 381 453 915 497	0.164 381 453 915 497
3	0.159 711 414 066 111	0.164 488 426 505 073	0.164 482 760 527 739
4	0.167 708 334 514 884	0.164 480 538 599 000	0.164 481 025 806 042
5	0.162 155 301 413 564	0.164 480 897 128 353	0.164 480 896 552 227
6	0.166 234 803 732 817	0.164 480 894 717 325	0.164 480 893 640 937
7	0.163 111 643 694 762	0.164 480 893 606 728	0.164 480 893 688 676
8	0.165 579 162 855 583	0.164 480 893 702 656	0.164 480 893 698 442
9	0.163 580 602 633 196	0.164 480 893 699 380	0.164 480 893 699 234
10	0.165 232 198 806 363	0.164 480 893 699 272	0.164 480 893 699 288
11	0.163 844 487 813 342	0.164 480 893 699 295	0.164 480 893 699 292
12	0.165 026 841 360 043	0.164 480 893 699 292	0.164 480 893 699 293
13	0.164 007 426 937 253	0.164 480 893 699 293	0.164 480 893 699 293
14	0.164 895 394 970 447	0.164 480 893 699 293	0.164 480 893 699 293
15	0.164 115 002 453 607	0.164 480 893 699 293	0.164 480 893 699 293
exact	0.164 480 893 699 293	0.164 480 893 699 293	0.164 480 893 699 293

TABLE VII. Evaluation of $10^{-1}\,\mathrm{Li}_3(0.99999)$ with the CNCT.

n	$\mathbf{S}_n$	$d_n^{(0)}\big(1, \mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1, \mathbf{S}_0\big)$
0	0.133 331 333 415 539	0.133 331 333 415 539	0.133 331 333 415 539
1	0.116 665 166 707 769	0.120 474 532 168 000	0.120 474 532 168 000
2	0.121 603 216 117 468	0.120 176 326 936 846	0.120 176 326 936 846
3	0.119 520 007 764 208	0.120 206 042 152 677	0.120 204 748 497 388
4	0.120 586 594 446 594	0.120 203 955 328 380	0.120 204 079 128 106
5	0.119 969 366 038 597	0.120 204 046 033 711	0.120 204 045 387 208
6	0.120 358 052 152 572	0.120 204 045 725 809	0.120 204 045 378 284
7	0.120 097 666 726 411	0.120 204 045 413 120	0.120 204 045 434 802
8	0.120 280 540 991 745	0.120 204 045 439 707	0.120 204 045 438 553
9	0.120 147 227 650 952	0.120 204 045 438 749	0.120 204 045 438 726
10	0.120 247 386 435 540	0.120 204 045 438 729	0.120 204 045 438 733
11	0.120 170 239 824 481	0.120 204 045 438 733	0.120 204 045 438 733
12	0.120 230 916 813 857	0.120 204 045 438 733	0.120 204 045 438 733
13	0.120 182 336 147 846	0.120 204 045 438 733	0.120 204 045 438 733
14	0.120 221 833 436 597	0.120 204 045 438 733	0.120 204 045 438 733
15	0.120 189 289 161 290	0.120 204 045 438 733	0.120 204 045 438 733
exact	0.120 204 045 438 733	0.120 204 045 438 733	0.120 204 045 438 733

TABLE VIII. Evaluation of $10^4\,\Phi(0.99999, 2, 10000)$ with the CNCT.

n	$\mathbf{S}_n$	$d_n^{(0)}\big(1, \mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1, \mathbf{S}_0\big)$
0	1.152 086 970 131 424	1.152 086 970 131 424	1.152 086 970 131 424
1	0.576 093 485 065 712	0.806 478 876 912 452	0.806 478 876 912 452
2	0.960 055 803 546 361	0.797 618 192 129 198	0.797 618 192 129 198
3	0.672 109 050 515 104	0.798 663 645 011 412	0.798 596 144 946 064
4	0.902 446 455 721 367	0.798 581 028 864 897	0.798 586 253 716 867
5	0.710 515 275 487 446	0.798 585 227 987 408	0.798 585 188 634 170
6	0.875 013 443 138 958	0.798 585 145 208 936	0.798 585 140 857 888
7	0.731 090 035 137 739	0.798 585 138 553 527	0.798 585 139 249 075
8	0.859 010 851 725 411	0.798 585 139 276 063	0.798 585 139 237 667
9	0.743 892 107 147 896	0.798 585 139 218 618	0.798 585 139 229 500
10	0.848 536 431 310 159	0.798 585 139 219 617	0.798 585 139 222 908
11	0.752 620 788 733 222	0.798 585 139 223 310	0.798 585 139 222 175
12	0.841 150 625 351 432	0.798 585 139 222 720	0.798 585 139 222 491
13	0.758 951 478 583 304	0.798 585 139 222 444	0.798 585 139 222 559
14	0.835 664 028 102 224	0.798 585 139 222 555	0.798 585 139 222 550
15	0.763 752 250 680 046	0.798 585 139 222 555	0.798 585 139 222 548
16	0.831 428 053 244 539	0.798 585 139 222 546	0.798 585 139 222 548
17	0.767 517 561 053 133	0.798 585 139 222 548	0.798 585 139 222 548
18	0.828 059 092 035 324	0.798 585 139 222 548	0.798 585 139 222 548
19	0.770 549 625 376 190	0.798 585 139 222 548	0.798 585 139 222 548
20	0.825 315 796 529 872	0.798 585 139 222 548	0.798 585 139 222 548
exact	0.798 585 139 222 548	0.798 585 139 222 548	0.798 585 139 222 548

TABLE IX. Evaluation of $10^{-4}\,{}_3F_2(1,3/2,5;9/8,47/8;0.99999)$ with the CNCT.

n	$\mathbf{S}_n$	$d_n^{(0)}\big(1,\mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1,\mathbf{S}_0\big)$
0	0.343 961 195 195 881	0.343 961 195 195 881	0.343 961 195 195 881
1	0.172 030 597 597 940	0.240 789 533 227 428	0.240 789 533 227 428
2	0.286 614 046 845 766	0.238 145 631 122 015	0.238 145 631 122 015
3	0.200 705 485 068 072	0.238 457 646 856 530	0.238 437 505 168 542
4	0.269 409 131 416 317	0.238 433 043 073 649	0.238 434 595 265 258
5	0.212 175 660 263 795	0.238 434 334 888 202	0.238 434 314 202 617
6	0.261 216 282 820 282	0.238 434 307 597 861	0.238 434 305 531 575
7	0.218 319 995 918 563	0.238 434 299 034 546	0.238 434 301 380 046
8	0.256 437 590 806 900	0.238 434 297 734 183	0.238 434 298 863 521
9	0.222 142 795 720 768	0.238 434 298 685 404	0.238 434 298 539 687
10	0.253 309 971 367 287	0.238 434 298 885 696	0.238 434 298 711 196
11	0.224 749 013 245 827	0.238 434 298 760 156	0.238 434 298 769 407
12	0.251 104 853 004 796	0.238 434 298 751 345	0.238 434 298 766 552
13	0.226 638 970 976 291	0.238 434 298 765 877	0.238 434 298 763 311
14	0.249 467 024 022 419	0.238 434 298 763 853	0.238 434 298 763 230
15	0.228 071 952 328 669	0.238 434 298 762 980	0.238 434 298 763 331
16	0.248 202 731 012 486	0.238 434 298 763 382	0.238 434 298 763 332
17	0.229 195 680 986 673	0.238 434 298 763 343	0.238 434 298 763 330
18	0.247 197 365 060 190	0.238 434 298 763 322	0.238 434 298 763 330
19	0.230 100 443 575 937	0.238 434 298 763 332	0.238 434 298 763 330
20	0.246 378 830 994 286	0.238 434 298 763 330	0.238 434 298 763 330
exact	0.238 434 298 763 330	0.238 434 298 763 330	0.238 434 298 763 330

TABLE X. Evaluation of $10^{-1}\,{}_3F_2(1,3,7;5/2,14;0.99999)$ with the CNCT.

n	$\mathbf{S}_n$	$d_n^{(0)}\big(1,\mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1,\mathbf{S}_0\big)$
0	0.354 205 299 194 014	0.354 205 299 194 014	0.354 205 299 194 014
1	0.227 102 649 597 007	0.268 438 401 594 236	0.268 438 401 594 236
2	0.288 360 357 978 268	0.266 943 489 834 494	0.266 943 489 834 494
3	0.254 808 733 179 764	0.267 121 290 068 426	0.267 112 224 310 036
4	0.274 632 798 536 574	0.267 100 105 730 057	0.267 101 775 442 210
5	0.262 289 292 919 201	0.267 103 299 447 369	0.267 102 948 107 378
6	0.270 285 887 617 646	0.267 102 742 191 670	0.267 102 815 189 177
7	0.264 938 303 793 252	0.267 102 836 668 825	0.267 102 824 285 564
8	0.268 610 033 794 438	0.267 102 822 243 079	0.267 102 823 985 155
9	0.266 031 550 394 689	0.267 102 824 198 445	0.267 102 823 985 352
10	0.267 878 112 865 182	0.267 102 823 960 840	0.267 102 823 984 751
11	0.266 532 664 737 038	0.267 102 823 987 269	0.267 102 823 984 758
12	0.267 528 211 474 989	0.267 102 823 984 510	0.267 102 823 984 761
13	0.266 781 286 870 185	0.267 102 823 984 786	0.267 102 823 984 762
14	0.267 348 762 560 130	0.267 102 823 984 759	0.267 102 823 984 762
15	0.266 912 657 747 879	0.267 102 823 984 762	0.267 102 823 984 762
16	0.267 251 339 082 039	0.267 102 823 984 762	0.267 102 823 984 761
17	0.266 985 765 115 182	0.267 102 823 984 761	0.267 102 823 984 762
18	0.267 195 878 973 722	0.267 102 823 984 762	0.267 102 823 984 762
19	0.267 028 262 510 152	0.267 102 823 984 762	0.267 102 823 984 762
20	0.267 163 009 854 165	0.267 102 823 984 762	0.267 102 823 984 762
exact	0.267 102 823 984 762	0.267 102 823 984 762	0.267 102 823 984 762

TABLE XI. Evaluation of $10^{-1}\,{}_3F_2(1,3,7;5/2,14;1)$ with the CNCT.

n	$\mathbf{S}_n$	$d_n^{(0)}\big(1,\mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1,\mathbf{S}_0\big)$
0	0.354 212 896 979 703	0.354 212 896 979 703	0.354 212 896 979 703
1	0.227 106 448 489 852	0.268 443 680 394 043	0.268 443 680 394 043
2	0.288 366 524 620 961	0.266 948 705 538 902	0.266 948 705 538 902
3	0.254 813 300 376 035	0.267 126 514 679 686	0.267 117 448 402 341
4	0.274 638 493 612 452	0.267 105 329 111 381	0.267 106 998 932 606
5	0.262 294 169 832 612	0.267 108 523 033 192	0.267 108 171 668 587
6	0.270 291 370 710 880	0.267 107 965 739 446	0.267 108 038 742 291
7	0.264 943 330 016 988	0.267 108 060 223 478	0.267 108 047 839 255
8	0.268 615 409 392 370	0.267 108 045 796 597	0.267 108 047 538 821
9	0.266 036 655 402 122	0.267 108 047 752 131	0.267 108 047 539 018
10	0.267 883 429 838 931	0.267 108 047 514 503	0.267 108 047 538 417
11	0.266 537 813 952 027	0.267 108 047 540 935	0.267 108 047 538 424
12	0.267 533 494 713 522	0.267 108 047 538 176	0.267 108 047 538 427
13	0.266 786 462 107 999	0.267 108 047 538 452	0.267 108 047 538 428
14	0.267 354 025 525 918	0.267 108 047 538 425	0.267 108 047 538 428
15	0.266 917 848 923 282	0.267 108 047 538 428	0.267 108 047 538 428
16	0.267 256 589 412 242	0.267 108 047 538 428	0.267 108 047 538 428
17	0.266 990 966 387 224	0.267 108 047 538 428	0.267 108 047 538 428
18	0.267 201 121 176 708	0.267 108 047 538 428	0.267 108 047 538 428
19	0.267 033 470 369 207	0.267 108 047 538 428	0.267 108 047 538 428
20	0.267 168 246 683 931	0.267 108 047 538 428	0.267 108 047 538 428
exact	0.267 108 047 538 428	0.267 108 047 538 428	0.267 108 047 538 428

TABLE XII. Evaluation of $10^{-5}\,\sum_{l=0}^{\infty}(2\,l+1)\,j_l(i\,0.9999\times 0.7)\,h_l^{(1)}(i\,0.7)$ with the CNCT.

n	$\mathbf{S}_n$	$d_n^{(0)}\big(1,\mathbf{S}_0\big)$	$\delta_n^{(0)}\big(1,\mathbf{S}_0\big)$
0	−0.206 084 520 894 668	−0.206 084 520 894 668	−0.206 084 520 894 668
1	−0.103 046 104 279 554	−0.144 259 660 091 669	−0.144 259 660 091 669
2	−0.171 733 352 076 042	−0.142 674 251 704 499	−0.142 674 251 704 499
3	−0.120 220 473 724 774	−0.142 861 066 165 942	−0.142 849 004 178 547
4	−0.161 427 969 242 195	−0.142 846 281 288 224	−0.142 847 217 919 030
5	−0.127 091 187 739 732	−0.142 847 117 647 583	−0.142 847 093 177 734
6	−0.156 520 814 934 640	−0.142 847 153 435 206	−0.142 847 135 794 481
7	−0.130 771 367 274 807	−0.142 847 152 002 732	−0.142 847 148 838 958
8	−0.153 658 281 433 404	−0.142 847 142 941 523	−0.142 847 145 983 994
9	−0.133 061 584 756 825	−0.142 847 142 048 135	−0.142 847 143 380 152
10	−0.151 784 408 324 881	−0.142 847 143 286 397	−0.142 847 143 026 940
11	−0.134 623 097 855 864	−0.142 847 143 324 758	−0.142 847 143 169 466
12	−0.150 463 209 081 729	−0.142 847 143 181 923	−0.142 847 143 211 999
13	−0.135 755 492 037 282	−0.142 847 143 200 611	−0.142 847 143 208 928
14	−0.149 481 826 233 342	−0.142 847 143 210 852	−0.142 847 143 207 036
15	−0.136 614 208 810 295	−0.142 847 143 206 780	−0.142 847 143 207 092
16	−0.148 724 109 578 581	−0.142 847 143 206 921	−0.142 847 143 207 139
17	−0.137 287 765 215 359	−0.142 847 143 207 226	−0.142 847 143 207 135
18	−0.148 121 427 644 456	−0.142 847 143 207 123	−0.142 847 143 207 135
19	−0.137 830 196 213 572	−0.142 847 143 207 132	−0.142 847 143 207 135
20	−0.147 630 649 333 917	−0.142 847 143 207 137	−0.142 847 143 207 135
21	−0.138 276 357 305 065	−0.142 847 143 207 134	−0.142 847 143 207 135
22	−0.147 223 291 777 104	−0.142 847 143 207 135	−0.142 847 143 207 135
23	−0.138 649 758 251 940	−0.142 847 143 207 135	−0.142 847 143 207 135
24	−0.146 879 773 958 865	−0.142 847 143 207 135	−0.142 847 143 207 135
25	−0.138 966 841 391 919	−0.142 847 143 207 135	−0.142 847 143 207 135
exact	−0.142 847 143 207 135	−0.142 847 143 207 135	−0.142 847 143 207 135